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SCIENCE DE

ALGEBRA

FOR

HIGH SCHOOLS AND COLLEGES:

CONTAINING

A SYSTEMATIC EXPOSITION AND APPLICATION

OF THE

ELEMENTARY AND HIGHER PRINCIPLES OF THE SCIENCE.

BY JAMES B. DODD, A. M.,
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PREFACE.

THE following work is designed to meet the wants of the student at the commencement, and during the continuance, of his Algebraic studies.

It aims at the most methodical *arrangement*, the clearest *expositions*, the best elementary *exercises*, the most varied and useful *applications*—in all these respects presenting some new features, which have been adopted as improvements in the method of teaching this science.

It contains all that has been deemed appropriate to an Algebraic treatise, in a general course of mathematical studies, or necessary in preparation for the higher works of the course. To the latter purpose it is believed to be particularly well adapted.

The appendage of a *Table of Logarithms*, and the introduction of logarithmic calculations, were considered necessary to a proper presentation of so important a subject; and will doubtless be generally regarded as enhancing the value of the work.

TRANSYLVANIA UNIVERSITY,
December 28*th*, 1853.

REMARKS

ON THE METHOD OF USING THIS WORK, AND CONDUCTING EXAMINATIONS IN ALGEBRA.

The following remarks may be useful to the less experienced Teacher using this work, who would make it fully efficient for the purposes intended.

1. The definitions and propositions numbered (1), (2), (3), &c., and the Rules I, II, III, &c., should be accurately memorized and recited by the Student.

2. The accompanying examples, illustrations, or demonstrations, should be required of the Student, and discussed with him on the part of the Teacher, with reference to the *principles* involved in them.

3. The *oral* exercises in the earlier parts of the work, should be exacted; and the Student should often be examined on the exercises under the Rules, with his book closed.

4. In the solution of Equations and Problems, he should explain each part of the operation, as exemplified in different parts of the work.

5. The Analysis of Contents (see the next page) will be convenient for reviews on the theory of the science; and such reviews should be frequent. The Student will thus become familiar with the phraseology, principles, and order of the science.

6. The Student's acquisitions will depend very much on the *exactness*, as well as on the frequency, with which he is examined. The requisitions made on him should be adapted to his capabilities,—which, it should be remembered, are liable to be sometimes *overrated*, and sometimes *underrated*, by Authors and Teachers.

ANALYSIS OF CONTENTS.

☞ This Analysis is designed to be used in oral examinations, in reviews. The Teacher will name the *topic* as presented in this table; the Learner will respond according to his knowledge of the subject.

For example: the Teacher will say, "Science and Art;" the Learner will respond, "Science is knowledge reduced to a system; Art is knowledge applied to practical purposes."

CHAPTER I.

Preliminary Definitions and Exercises.—Page 1 . . . 8.

Science and Art, (1).—A Unit—Numbers, (2).—Quantity—Whether Numbers are *quantities*, (3).—Mathematics—Its most general Divisions—Arithmetic—Geometry, (4).—Algebra, (5).—Symbols of Quantities, (6).—Symbols of Operations, the Sign + plus, (7).—The Sign − *minus*, (8).—The Sign × *into*—A *point* (.) between Quantities—Quantities in *juxtaposition*, (9).—The sign ÷ *by*—Division otherwise denoted—An Integral Quantity, (10).—Use of the Parenthesis or Vinculum, (11).—Factors and Constant Product, (12).—Powers of Quantities, (13).—Roots of Quantities, (14).—The Coefficient of a Quantity, (15).—The Exponent of a Quantity, (16).—What an Integral Coefficient indicates—an Integral Exponent, (17).—Similar and Dissimilar Quantities, (18).—An Algebraic Monomial, (19).—An Algebraic Polynomial—A Binomial—A Trinomial, (20).—Whether the Value of a Polynomial is affected by changing the Order of its Terms, (21).—A Polynomial arranged by *powers*, (22).—A Homogeneous Polynomial—Dimensions and Degrees of the Terms of a Polynomial, (23).—Positive and Negative Quantities, (24).—Effect of a Negative Quantity in an Expression or a Calculation—Occasional Use of the Positive and Negative Signs, (25)

CHAPTER II.

ADDITION.—SUBTRACTION.—MULTIPLICATION.—DIVISION.—9 . . . 26.

Algebraic ADDITION, (26).—How to Add Similar Terms with *like signs*, (27).—Sum of Two Equal Similar Terms with *contrary signs*, (28).—How to Add unequal Similar Terms with contrary signs, (29).—Sum of Two or more Dissimilar Terms, (30).

RULE I. *For the Addition of Algebraic Quantities*, (31).

Calculations on the same Polynomial in two or more ( )'s, (32).

Algebraic SUBTRACTION, (33).—How to Subtract a Monomial from another Quantity, (34).—How to Subtract a Monomial from a Dissimilar Quantity, (35).

RULE II. *For the Subtraction of Algebraic Quantities*, (36).

How to denote the Subtraction of a *negative* Monomial—of a Polynomial, (37).—Change of *signs* in a Polynomial without affecting its Value, (38).

Algebraic MULTIPLICATION—When the Multiplier is *positive*—When the Multiplier is *negative*, (39).—Product of Two Monomials, (40).—Exponent, in the Product, of a Letter occurring in both the Monomials multiplied together, (41).—*Sign* of the Product—*Reason* for this when both the Quantities are *negative*—When One of them is positive and the Other *negative* (42).—Product when Either of the Two Factors is 0, (43).

RULE III. *To Multiply a Monomial into a Polynomial*, (44).

RULE IV. *To Multiply a Polynomial into a Polynomial*, (45).

Algebraic DIVISION, (46).—How to find the Quotient of Two Monomials, (47).—Value of any Quantity with *exponent* 0, (48).—*Sign* of the Quotient, and Principle which determines it, (49)—Quotient of 0 divided by any Quantity, and of any Quantity divided by 0, (50).

RULE V. *To Divide a Monomial into a Polynomial*, (51).

RULE VI. *To Divide a Polynomial into a Polynomial*, (52).

How the *indicated Product* of Two or more Factors may be divided, (53).

CHAPTER III.

COMPOSITE QUANTITIES.—COMMON MEASURE.—COMMON MULTIPLE.—27 . . . 40.

A Composite and a Prime Quantity, (54).—Decomposition of a Quantity—Divisor and Quotient as Factors of a Quantity, (55).—What the Difference of Two Quantities will divide, (56).—What the sum of two Quantities will divide, (57).—Product of the Sum and Difference of two Quantities, (58).—Square of the Sum of two Quantities, (59).—Square of the Difference of two Quantities, (60).—Case in which a Trinomial may be resolved into Two unequal Binomial Factors, (61.)—One Quantity a MEASURE of another—A Common Measure of Two or more Quantities, (62).—Greatest Common Measure of Two or More Quantities, (63).—Composition of the Greatest Common Measure, (64).—Principle on which depends the Rule for the Greatest Common Measure, (65).

RULE VII.—*To Find the Greatest Common Measure of Two Quantities*, (66).

Whether the *Signs* in a Common Measure may be changed, (67).—Of a Factor which is common to all the Terms of the Dividend, (68).—Of a Factor which is contained in all the Terms of the two Polynomials, (69).—One Quantity a MULTIPLE of another.—A Common Multiple of Two or more Quantities, (70).—Least Common Multiple of Two or More Quantities, (71).—Composition of the Least Common Multiple, (72).—Relation of Least Common Multiple to Product and Greatest Common Measure, (73).

RULE VIII. *To Find the Least Common Multiple of Two or More Quantities*, (74).

CHAPTER IV.

FRACTIONS.—41 . . . 66.

An Algebraic FRACTION, (75).—Of a Quantity with a *negative* Exponent, (76).—Two Methods of representing the Quotient when the Divisor is not a Factor of the Dividend, (77).—How any Factor may be transferred from the

Numerator to the Denominator, and *vice versa*, (78).—*Reciprocal* of a Quantity, (79).—Reciprocal of a Fraction, (80).—Constant Value of a Fraction, (81).—When a Fraction is *positive*, and when *negative*, (82).—Signification of + or − prefixed to a Fraction, (83).—Changes of *Signs* without affecting the Value of a Fraction—How a Polynomial may be changed from Positive to Negative, (84).—A Fraction reduced to *lower* Terms—How a *binomial* Common Measure may often be discovered, (85).

Rule IX. *To Reduce a Fraction to its Lowest Terms*, (86).

When Two or more Fractions are said to have a Common Denominator. —How Fractions may be reduced, mentally, to a Common Denominator, (87).

Rule X. *To Reduce Two or More Fractions to a Common Denominator*, (88).

An Integral Quantity, (89).—A Mixed Quantity, (90).—An Improper Fraction, (91).

Rule XI. *To Reduce an Integral or a Mixed Quantity to an Improper Fraction*, (92).

Rule XII. *To Reduce an Improper Fraction to an Integral or a Mixed Quantity*, (93).

By what means the Sum of Two or More Fractions is found, (94).

Rule XIII. *For the Addition of Fractions*, (95).

By what means the Difference of Two Fractions is found, (96).

Rule XIV. *For the Subtraction of Fractions*, (97).

Product of Two or more Fractions, (98).—Effect of Multiplying by a Fraction, (99).—Compound Fractions, (100).—Equivalent of Multiplying Two or more Fractions together, (101).

Rule XV. *For the Multiplication of Fractions*, (102).

How a Fraction is Multiplied by its own Denominator, and what Cancellations may be made in the Multiplication of Fractions, (103).—Quotient of Two Fractions, (104)—Complex or Mixed Fractions, (105).

Rule XVI. *For the Division of Fractions*, (106).

CHAPTER V.

SIMPLE EQUATIONS.—67 . . . 90.

An EQUATION—The First Member—The Second Member, (107).—For what purposes Equations are employed—How applied to the Solution of Questions, (108).—The Solution of an Equation—Verification of the Value found for the Unknown Quantity, (109).—A *Simple* Equation—A Quadratic Equation—A Cubic Equation, (110).—A Numerical Equation—A Literal Equation—An Identical Equation, (111).—Transformation of an Equation, (112).—An *Axiom*—Axiom *first*, *second*, &c., (113).—How the Value of the Unknown Quantity is found—Transformations necessary, (114).—How to clear an equation of Fractions—How by means of Least Common Multiple—Advantage of this Method, (115).—How any term may be Transposed from one Side of an Equation to the other, (116).—Change of the Signs in an Equation, (117).

RULE XVII. *For the Solution of a Simple Equation containing but one unknown quantity*, (118.)

A Problem, and in what its Solution consists—General Method of forming the Equation of a Problem, (119).—Solution of Problems with Two or more Unknown Quantities—Independent Equations, (120).—General Method of solving Two Equations, (121).—Elimination by Addition or Subtraction, (122).—Elimination by Substitution, (123).—Elimination by Comparison, (124).—Solution of Three Equations—Of Four Equations, (125).—Of Problems in which there are Three or more Required Quantities, (126).

CHAPTER VI.

RATIO.—PROPORTION.—VARIATION.—91. . . . 110.

Ratio of one Quantity to another, (127).—*Sign* of Ratio, (128).—How the value of a Ratio may be represented, (129).—Direct and *Inverse* Ratio, (130).—Compound Ratio, (131).—Ratio of the *first* to the *last* of any Number of Quantities, (132).—Duplicate and Triplicate Ratios, (133).—Equimultiples and Equisubmultiples, (134).—Ratio of Equimultiples and Equisubmultiples, (135).—PROPORTION, (136).—Four Quantities in Proportion, (137).—Three quantities in Proportion, (138).—Direct and *Inverse* Proportion, (139).—*Sign* of Proportion, (140).—Inverse Converted into Direct Pro portion, (141).—VARIATION—Variation *direct*—Variation *inverse*, (142).—Product of Two Quantities varying *inversely* with each other, (143).—Variation, an Abbreviated Proportion, (144).—A *Theorem*—A Corollary, (145).—Ratio of two Fractions having a *common* Term, (146).—How the value of a Fraction *varies*, (147).—Corresponding Equalties and Inequalities between the Antecedents and Consequents of a Proportion, (148).—A Proportion converted into an Equation, (149).—A Fourth Proportional, how found, (150).—Product of the Extremes when Three Quantities are in Proportion, (151).—Mean Proportional, how found, (152).—An Equation converted into a Proportion, (153).—Ratio of the *first* to the *third* of Three Proportional Quantities, (154).—Proportion by Inversion, (155).—Proportion by Alternation, (156). What Multiplications may be made in a Proportion, (157).—What Divisions may be made in a Proportion, (158).—Proportion by Composition, (159).—Proportion by Division, (160).—Proportion between the Sum of two or more Antecedents and that of their Consequents, (161).—A Proportion derived from Two other Proportions in which there are *common* Terms, (162).—Proportion between the Sums and Differences of the Antecedents and Consequents, (163).—Products of the Corresponding Terms of Two or more Proportions, (164).—Proportion between the Powers or Roots of Proportional Quantities, (165).—Substitution of Factors in a Proportion, (166).—*General Solution* of a Problem, (167).—Two Numbers found from their Sum and Difference, (168).—An Algebraic *Formula*, (169).—Of a Proportion occurring in the Solution of a Problem, (170).—*Percentage*—Ratio of Percentage—Basis of Percentage, (171).—Amount of Percentage, how found, (172).—*Interest*—The Principal—The Amount, (173).—Amount of Interest, how found, (174).

CHAPTER VII.

ARITHMETICAL, HARMONICAL, AND GEOMETRICAL PROGRESSION.—111...120.

An Arithmetical PROGRESSION, (175).—Last Term of an Arithmetical Progression, equal to what, (176).—Common Difference of the Terms, (177).—Sum of the two Extremes, (178).—Arithmetical Mean, (179).—Sum of all the Terms, (180).—*Formulas* in Arithmetical Progression, (181).—A Harmonical PROGRESSION, (182).—An Harmonical converted into an Ari metical Progression, (183).—Harmonical Mean, (184).—A Geometerical PROGRESSION, (185).—Last Term of a Geomitrical Progression, (186).—Power of the *ratio* found from the First and last Terms, (187).—Product of the two Extremes, (188).—Geometrical Mean, (189).—Sum of all the Terms, (190) —Sum of an *infinite* number of Decreasing Terms, (191).—*Formulas* in Geometrical Progression, (192).

CHAPTER VIII.

PERMUTATIONS AND COMBINATIONS.—INVOLUTION.—BINOMIAL.—THEOREM. —EVOLUTION.—121. . . . 146.

PERMUTATIONS, (193).—Number of Permutations, how found, (194).—COMBINATIONS, (195).—Number of Combinations, how found, (196).—INVOLUTION, (197).—A Higher Power found from lower Powers of the same Quantity, (198.)—Powers of *unity*, (199).—Powers of Monomials, how found, (200).—Powers of Fractions, how found—Powers of a *mixed* Quantity, (201). *Sign* to be Prefixed to a Power, (202).—Powers of Binomials, or of any Polynomials, (203).—BINOMIAL THEOREM—*Exponents* in any Power of $(a \pm b)$ —*Coefficients*—*Signs*, (204).—Formula for the development of $(a+b^n)$—At what Term the development will *terminate*, (205).—EVOLUTION—Extracting the Square Root, in what it consists—The Cube Root, (206).—Roots of *unity*, (207).—Roots of Monomials, how found, (208).—Roots of Fractions, how found—Roots of a *mixed* Quantity, (209).—Of a Root whose Exponent is resolvable into two Factors, (210).—What denoted by the Numerator and Denominator of a Fractional Exponent, (211).—Root of a power of a Quantity, (212).—Equivalent Exponents, (213).—*Sign* to be Prefixed to an *odd* Root of a Quantity, (214).—Sign to be Prefixed to an *even* Root, (215).—Of an *even* Root of a *negative* Quantity, (216).—How the Roots of Polynomials may be discovered, (217).

RULE XVIII. *To Extract the Square Root of a Polynomial*, (218.)

Principle for determining the Number of Figures in the Square Root of a Number, (219).—Square of any two Parts into which a number may be divided, (220).—Periods to be formed in Extracting the Square Root of a Decimal Fraction —Why the last Period must be *complete*—Number of Decimal Figures to be made in the Root, (221).

RULE XIX. *To Extract the Cube Root of a Polynomial*, (222.)

Principle for determining the Number of Figures in the Cube Root of a Number, (223).—Cube of any two Parts into which a Number may be divided, (224).

RULE XX. *To Extract the Cube Root of a Number*, (225).

Periods to be formed in extracting the Cube Root of a Decimal Fraction —Why the last period must be *Complete*—Number of decimal Figures to be made in the Root,(226).—How any Root whatever of a Polynomial might be extracted, (227).

RULE XXI. *To Extract any Root of a Polynomial*, (228).

CHAPTER IX.

IRRATIONAL OR SURD QUANTITIES.—IMAGINARY QUANTITIES.—147. . .166.

Perfect and Imperfect *Powers*, (229).—A *Rational* Quantity—An Irrational or *Surd* Quantity—*Radical* Quantities, (230).—Radical Sign—How this Sign may always be superseded, (231).—Similar and Dissimilar Surds,(232). How a Rational Quantity may be expressed under the Form of a Surd, (233). Transfer of an Exponent between Factors and Product, (234).—To what the Exponent of a Quantity may be changed, (235).—Product of a Rational and an Irrational Factor, (236).—How a Surd may be *simplified*, (237).—How a Fractional may be reduced to an *Integral* Surd, (238).—Surds of Different Roots reduced to the Same Root, (239).—How to find the *Sum* or Difference of Similar Surds—of *Dissimular* Surds, (240).—How to find the *Product* or Quotient of Surds of the *same root*—of *different* Roots—of any two Roots of the *Same* Quantity, (241).—Expediency of *rationalizing* a Surd Divisor or Denominator, (242).—How a Monomial Surd may be made to produce a Rational Quantity—a *binomial* Surd—a *trinomial* Surd, (243).—Of the Powers and Roots of Irrational Quantities, (244).—Of the Square Root of a Numerical Binomial of the form $a \pm \sqrt{b}$ (245).—Of *Imaginary* Quantities—From what an Imaginary Quantity results,(246).—Of the Calculus of Imaginary Quantities—By what means the Sign affecting the Product of two Imaginaries may be determined, (247).—Resolution of an Imaginary Quantity, (248).—Product of two Imaginary *Square* Roots, (249)

CHAPTER X.

QUADRATIC AND OTHER EQUATATIONS.—167 . . . 204.

A Quadratic Equation—A Cubic Equation—A Biquadratic Equation, (250).—A Pure Equation—An Affected and a Complete Equation, (251).—A *Root* of an Equation, (252).—Principle for determining a Divisor of an Equation, (253).—Principle for determining a Root of an Equation, (254). Number of Roots of an Equation—Whether the several Roots of an Eqaution are necessarily *unequal*, (255).

RULE XXII. *For the Solution of a Pure Equation*, (256).

How a *Surd* Quantity in an Equation may be *rationalized*, (257).—How two Surds in an Equation may be rationalized, (258).—Of an Equation containing a Fraction whose terms are both *irrational*, (259).

RULE XXIII. *For the Solution of a Complete Equation of the Second Degree*, (260).

How any Binomial of the form ax^2+bx may be made a *Perfect Square* (261).

RULE XXIV. *To Reduce an Equation of the form* ax^2+bx *to a Simple Equation*, (262).

Of Equations having a Quadratic Form with reference to a Power or Root of the Uuknown Quantity, (263).—Of the Solution of two Equations, one or both of the Second or a Higher Degree, Containing two Unknown Quantities, (264).—Solutions by means of an Auxiliary Unknown Quantity, (265).—Solutions by means of two Auxiliary Unknown Quantities, (266). —General Law of the Coefficients of Equations, (267).—Determination of the Integral Roots of Equations, (268).—Solution of Equations by *Approximation*, (269).—General Method of Elimination,(270).

CHAPTER XI.

GENERAL DESCRIPTION OF PROBLEMS.—MISCELLANEOUS PROBLEMS.—205 . . 217.

A Determinate Problem—How a Determinate Problem is represented, (270).—An Indeterminate Problem—How represented, (271).—An Impossible Problem—How represented, (272).—Greatest Product of any two Parts into which any Quantity may be divided, (273).—Signification of the Different Forms under which the value of the Unknown Quantity may be found in an Equation, (274).—*Inequations*, (275).—For what purpose Inequations are employed, (276).—Inequalities in the *same*, and in a *contrary sense*, (277).—Inequalities between *negative* quantities, (278).—Transformation of Inequations, (279).

CHAPTER XII.

GENERAL THEORY OF EQUATIONS.—STURM'S THEOREM.—HORNER'S METHOD OF SOLVING THE HIGHER EQUATIONS.—221 ... 244.

A General Theory of Equations, (280).—General Equation—If any of the lower powers of x be wanting in the given Equation, (281).—Change of Signs of the Roots of Equations, (282).—How the *positive* roots become *negative*, and conversely, (283).—*Functions* of variable Quantities, (284).—Derivative Functions—First, second, and third Derivatives—Development of the function X after $x+h$ has been substituted for x, (285).—Law of Derivative Functions, (286).—Derivatives of Composite Functions, (287).—Equal Roots of Equations, (288).—If an Equation be divided by the Product of the binomial Factors corresponding to its *equal roots*, (289).—Real Roots of Equations, (290).—Criterion for the *integral part* of a root of an Equation, (291).—Irrational and Imaginary Roots, (292).—Imaginary Roots of an Equation of an *even* degree, (293).—Variations and Permanences of Signs, (294).—STURM'S THEOREM—If the Equation has *equal roots*—Statement of the Theorem, (295).—Limits and Situation of the Real Roots of Equations, (296).—HORNER'S METHOD of solving the Higher Equations, (297).

RULE XXV.—*To find the Real Roots of the Higher Equations*, (298).

Recapitulatory Remarks on the Higher Equations, (299).

CHAPTER XIII.

CONTINUED FRACTIONS.—APPROXIMATING FRACTIONS.—245 ... 250.

A Continued Fraction—For what purpose Continued Fractions are chiefly useful—*Integrant* Fractions, (300).—Relative values of the successive approximations, (301).—How the successive approximations may be ascertained, (302).—Comparative value of any particular approximating Fraction, (303).—The same more definitely stated, (304).

CHAPTER XIV.

LOGARITHMS AND THEIR APPLICATIONS.—251 ... 276.

Logarithms of Numbers—Whether *unity* can be the *base* of a system of Logarithms, (305).—The Common system of Logarithms—For what purpose they are used—*Naperian* Logarithms, (306).—Logarithm of unity, and of the *base* of Logarithms, (307).—Logarithm of a Product, (308).—Logarithm of a Quotient, (309).—

Logarithm of a Power or Root, (310).—Logarithm of 0, (311).—Progression of Logarithms, (312).—Computation of Logarithms, (313).—Logarithmic Series—*Formula* for computing the Logarithm of $(n+1)$, (314).—*Modulus* of Naperian Logarithms, (315).—Modulus of the common system—How the Common Logarithm of a Number may be found, (316).—Ratio between the Logarithms of the same number in different systems, (317).—Naperian *base* of Logarithms, (318)—The *Characteristic* of a Logarithm, (319).—Logarithm of a Decimal Fraction, (320).—Practical rule for determining the Characteristic, (321).—The *decimal part* of the Logarithm of any number, (322).—TABLE of Logarithms, (323).—To find from the Table the Logarithm of a Number containing more than four figures, (324).—To find from the Table the number corresponding to any given Logarithm, (325).—To Multiply two or more Numbers together, by Logarithms, (326).—To Divide one Number by another, (327).—To raise any power of a number, (328).—To extract any Root of a Number, (329).—Arithmetical Complement of Logarithms—For what purpose the Arithmetical Complement is used, (330).—Compound Interest—Formula for the Amount, at Compound Interest, of the sum P for n years, if r denote the interest of $1, for 1 year, (331).—Problems on Population, (332).—An *Annuity*—a Perpetuity—an Annuity in possession—an Annuity in reversion—an Annuity in arrears, (333).—Formula for the Amount of the Annuity P, at compound Interest, for n years, if r denote the interest of $1 for 1 year, (334).—Formula for the Amount of the sum P annually put at Compound Interest, for n successive years, (335).—Present value of an Annuity, (336).—Present value of a Perpetuity, (337).—Present value of an Annuity in *reversion*, (338).—Present value of a Perpetuity P, in reversion for t years, (339).

CHAPTER XV.

INFINITE AND INFINITESIMAL QUANTITIES.—VANISHING FRACTIONS.—277 ... 284.

An Infinite Quantity—An Infinitesimal Quantity—Whether all Infinite Quantities, or all Infinitesimals are equal, (340).—Of 0 as representing an Infinitesimal, (341).—Any Finite quantity × Infinity, (342).—A Finite quantity × an Infinitesimal, (343).—A Finite quantity ÷ Infinity—A Finite quantity connected by + or − to Infinity, (344).—Infinity ÷ any Finite quantity, (345).—An Infinitesimal × Infinity, (346).—An Infinitesimal ÷ any Finite quantity, (347).—A Finite quantity ÷ an Infinitesimal, (348).—Infinity ÷ Infinity, or an Infinitesimal ÷ an Infinitesimal, (349).—An Infinitesimal with a Finite *exponent*, and with an Infinitesimal exponent, (350).—A Vanishing Fraction, (351).—Value of a Fraction which takes the form $\frac{0}{0}$ in consequence of a *Common* Factor in its two terms—Elucidation of the nature of a Vanishing Fraction, (352).—Different values of $\frac{0}{0}$, (353).

CHAPTER XVI.

INFINITE SERIES.—DIFFERENTIAL METHOD OF SERIES.—INTERPOLATION OF SERIES.—
285 ... 306.

An Infinite Series—The General Term of a Series—A *converging* Infinite Series—A diverging Infinite Series, (354).—Expansion of Series—The Algebraic methods of Expanding Series, (355).—Series by the Binomial Theorem—When the exponent of the Binomial is a *positive integer*—When it is a *proper* Fraction, or is *negative*, (356).—Series by Assumed Coefficients, (357).—Theorem on which this method of expanding Series depends, (358).—Separation of a Fraction into Component Partial Fractions, (359).—Summation of Infinite Series, (360).—Differential Method of Series (361).—To find the *n*th Term of a Series, (362).—To find the sum of *n* Terms of a Series, (363).—Interpolation of Series, (364).—Differential Method of Interpolating a Series, (365).—When the Term to be interpolated is one of the Equidistant Functions, (366).

A L G E B R A.

CHAPTER I.

PRELIMINARY DEFINITIONS AND EXERCISES.

Science and Art.

(1.) SCIENCE is knowledge reduced to a *system.*—ART is knowledge applied to practical purposes.

The Rules of Art are founded on the Principles of Science.

Numbers.—Quantity.

(2.) A UNIT is any thing regarded simply as *one;* and *numbers* are repetitions of a *unit.*

Thus the numbers *two, three,* &c., are repetitions of the unit *one.*

(3.) QUANTITY is any thing which admits of being measured.

Thus a *line* is a quantity, and we express its measure in saying it is so many *feet* or *inches* long. *Time, weight,* and *distance* are also quantities.

Numbers are quantities; for every number expresses the measure of itself in *units;* and numbers are used to express the measures of all other quantities. Thus we express the measure of Time by a number of *days, hours,* &c

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MATHEMATICS.

(4.) MATHEMATICS is the science of *quantity.* Its most general divisions are Arithmetic and Geometry.

ARITHMETIC is the science of *numbers;* or, when practically applied, the art of Calculation.

GEOMETRY is the science which treats of Extension—in *length*, *breadth*, and *height*, depth, or thickness.

(5.) ALGEBRA is a method of investigating the relations of both Arithmetical and Geometrical quantities, by *signs* or *symbols*.

Symbols of Quantities.

(6.) Quantities are represented in Algebra by *letters,—known* quantities usually by the *first*, and *unknown* or required quantities by the *last* letters of the Alphabet.

Thus the quantity a or b, that is, the quantity *represented* by a or b, will generally be understood as *known in value;* while the quantity x or y will be *unknown* or required.

Quantities represented by *letters* are called *literal* quantities, in contradistinction to *numbers* or numerical quantities.

Symbols of Operations.

(7.) The sign + *plus* prefixed to a quantity, denotes that the quantity is to be *added*, or taken *additively*.

Thus $a+b$, a plus b, denotes that the quantity b is to be added to the quantity a.

(8.) The sign — *minus* prefixed to a quantity, denotes that the quantity is to be *subtracted*, or taken *subtractively*.

Thus $a-b$, a minus b, denotes that the quantity b is to be subtracted from the quantity a.

(9.) The sign × *into* between two quantities, denotes that the two quantities are to be *multiplied* together.

Thus $a\times b$, a into b, denotes that the two quantities a and b are to be multiplied together.

A *point* (.) between two literal quantities, or a numerical and a literal quantity, also denotes that the two quantities are to be multiplied together.

$a.b$ denotes a into b, and $3.a$ denotes 3 into a, or 3 times a.

The Product of *several numbers* may be denoted by points betwe *them; thus* 1.2.3 denotes 1 *into* 2 *into* 3, the same as $1\times2\times3$.

Quantities in juxtaposition, without any sign between them, are to be multiplied together. Thus ab denotes a and b multiplied together; and axy denotes a, x, and y multiplied together.

(10.) The sign $\div$ *by* between two quantities, denotes that the quantity before the sign is to be *divided* by the one after it.

Thus $a\div b$, a by b, denotes that a is to be divided by b.

Division is also denoted by placing the *dividend over the divisor*, with a line between them, after the manner of a Fraction;

$\frac{a}{b}$ denotes a divided by b, the same as $a\div b$.

An *integral quantity*, in Algebra, is one which does not express any operation in *division*, whatever may be the *numerical values* which the letters represent.

(11.) A *parenthesis* ( ) enclosing an algebraic expression, or a *vinculum* $\overline{\quad\quad}$ drawn over it, connects the *value of that expression* with the sign which immediately precedes or follows it.

Thus $(a+b).c$, or $(a+b)c$, a plus b in a parenthesis *into* c, denotes that the *sum* of a and b is to be multiplied into c.

The same thing would be denoted by $\overline{a+b}\times c$, a plus b under a vinculum *into* c.—In $a+bc$, only b would be multiplied into c.

The *vinculum*, and the expression affected by it, are sometimes set *vertically*.

Thus $\left.\begin{array}{r} a \\ +b \\ -c \end{array}\right| x$ is equivalent to $(a+b-c)\,x$

or $\overline{a+b-c}\times x$.

PRELIMINARY EXERCISES.

In the elementary *oral* Exercises which are occasionally inserted, the Student should write down the *quantities* as they are read to him.

☞ Suppose the letters a, b, c to represent the numbers 3, 4 5, respectively; then what is the *numerical* value of $a+b-c$?

What is the value of $ab+c$? Of $abc-bc+a$?

What is the value of $ac\div b$? Of $ab+b-ac$?

What is the value of $bc\div a$? Of $abc+ac-bc$?

What is the value of $(a+b)\,c$, a plus b in a *parenthesis* into c?

What is the value of $(ab+c)\,a$? Of $(a+b+c)\,c$?

What is the value of $(a+b)\div c$? Of $(ab+b-c)\,a$?

What is the value of $(bc-a)\div b$? Of $(bc-a+b)\,b$?

The Teacher may propose other Exercises of the same nature, should he deem it necessary.

Factors.—Constant Product.

(12.) Two or more quantities multiplied together, are called the *factors* of their product; and the Product is the *same in value*, in whatever order its factors are taken.

Thus a and x are the *factors* of the product ax; and this product is the same in value as xa. So a, b, and c are the factors of the product abc, or acb, or bca, &c.

It is most convenient to set *literal* factors according to the order of the same letters in the Alphabet; thus ax; abc.

To understand why the product ax is equal to xa, consider a and x as representing *numbers*, and that the product of two numbers is the same, when either of them is made the *multiplier*.

For example, 25 *times* 7 is equal to 7 *times* 25. For 25 times 7 must be 7 times as many as 25 times 1, which is 25; that is, 25 *times* 7 is equal to 7 *times* 25.

☞ Prove that 14 times 9 is equal to 9 times 14.
Prove that 31 times 11 is equal to 11 times 31.
Prove that 23 times 15 is equal to 15 times 23.
Prove that 47 times 18 is equal to 18 times 47.

Powers and Roots.

(13.) The *first power* of a quantity is the *quantity itself;* thus the first power of 5 is 5, and the first power of a is a.

The *second power*, or *square*, of a quantity, is the product of the quantity *multiplied into itself*. Thus the second power, or *square*, of 5 is 5×5, which is 25; and the second power of a is aa.

The *third power*, or *cube*, of a quantity is the product of the quantity *multiplied into its second power*, or square. Thus the third power, or cube of 5 is $5 \times 5 \times 5$, which is 125; and the third power of a is aaa.

☞ What is meant by the *fourth power* of a quantity? What is meant by the *fifth power* of a quautity? By the *seventh power* of a quantity?

What is the *square* of 3? The *cube* of 4? The fourth power of 2? The *square* of 7? The cube of 6? The fourth power of 10?

(14.) The *second root*, or square root, of a quantity, is that quantity whose *square is equal to the given quantity.* Thus the square root of 9 is 3; and the square root of aa is a.

The *third root*, or cube root, of a quantity, is that quantity whose *third power*, or *cube*, is equal to the given quantity. Thus the cube root of 8 is 2; and the cube root of aaa is a.

☞ What is meant by the *fourth root* of a quantity? What is meant by the *fifth root* of a quantity? By the *ninth root* of a quantity?

What is the square root of 16? The cube root of 27? The fourth root of 16? The square root of 81? The cube root of 1000?

What is the *square* of the square *root* of 4? The cube of the cube root of 125? The square of the cube root of 64? The cube of the square root of 16?

Coefficients and Exponents.

(15.) The *coefficient* of a quantity is any *multiplier* prefixed to that quantity.—In a more general sense, the coefficient of a quantity is any *factor* forming a product with that quantity.

Thus, in $3a$, 3 is the coefficient of a, and denotes 3 times a. In $5ax$, 5 is the coefficient, denoting 5 times ax. In $\frac{1}{2}x$, $\frac{1}{2}$ is the coefficient of x, and denotes *one-half* of x.

When *no numerical* coefficient is prefixed, a *unit* is always to be understood. Thus a is $1a$, once a, and ax is $1ax$, once ax.

(16.) The *exponent* of a quantity is an *integer* annexed to it, to denote a *power*, or a fraction annexed to denote a *root*, of that quantity.

Thus a^2, a with exponent 2, denotes the second *power*, or square of a; x^3 denotes the third power, or cube, of x; and so on.

The *fractional exponents* $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and so on, denote, respectively the *square root*, cube root, &c., of the quantity to which they are annexed.

$a^{\frac{1}{2}}$, a with exponent $\frac{1}{2}$, denotes the square root of a; $x^{\frac{1}{3}}$ denotes the cube root of x; and so on.

When *no exponent* is annexed to a quantity, a *unit* is always to be understood. Thus a is a^1, the first power of a. (13.)

An exponent *is assigned to the product* of two or more factors, by affecting such product with a parenthesis, or a vinculum, and the exponent.

Thus $(ax)^2$ or $\overline{ax}^2$, ax in a parenthesis, or under a vinculum, with exponent 2, denotes the square of the product ax; whereas ax^2 denotes a into the *square* of x.

(17.) An *integral coefficient* indicates the repeated *addition* of a quantity to itself; while an integral *exponent* indicates the repeated *multiplication* of a quantity into itself.

Thus $3a$, 3 times a, is equivalent to $a+a+a$; while
a^3, the third power of a, is equivalent to aaa.

Coefficients and exponents are thus employed to abbreviate the language of Algebra.

☞ What is the *equivalent*, in Addition, of $2x$? Of $3ax$? Of $2abc$? Of $5a^2$? Of $4ay^2$? Of $3axy$?

What is the *equivalent*, in Multiplication, of a^2? Of x^3? Of ax^2? Of a^2x? Of abx^3? Of ac^2x^2?

Allowing the value of a to be 4, what is the value of a^2? Of $a^{\frac{1}{2}}$? Of a^3? Of $5a^2$? Of $10a^{\frac{1}{2}}$? Of $a^{\frac{1}{2}}a^3$? Of $(4a)^{\frac{1}{2}}$?

Allowing a to be 4, and b 9, what is the value of $ab^{\frac{1}{2}}$? Of $a^{\frac{1}{2}}b$? Of $a^2b^{\frac{1}{2}}$? Of $(ab)^{\frac{1}{2}}$? Of $a^{\frac{1}{2}}b^{\frac{1}{2}}$? Of $\frac{1}{2}ab$? Of $\frac{1}{3}(ab)^{\frac{1}{2}}$?

Similar and Dissimilar Quantities.

(18.) *Similar* quantities are such as have all the *literal factors*, with their respective *exponents*, the *same in each;* otherwise, the quantities are *dissimilar*.

Thus the two quantities $2ax^2$ and $5ax^2$ are *similar;* while $3ax^2$ and $4a^2x$ are dissimilar, the literal factors not having the same exponents in each.

☞ Are the two quantities ab and $3ab$ similar, or *dissimilar?* Are $4a$ and $3x$ similar, or dissimilar? Are ay and $3ya$ similar, or dissimilar? $5ax^{\frac{1}{2}}$ and $7a^{\frac{1}{2}}x$? abc^2 and $5bc^2$? $2ab$ and $2ab^2$? $3axy^2$ and ay^2x? $2bc^3$ and $3bc^{\frac{1}{2}}$?

Give an example of *three similar quantities.*—Give an example of three dissimilar quantities.—Another example of three similar quantities.—Another example of three dissimilar quantities.

Monomials and Polynomials.

(19.) An algebraic *monomial* is a symbol of quantity not composed of parts connected by the sign + or −.

Thus $3a$, $5ax$, $2x^2$, and $\frac{1}{2}abc^3$ are monomials.

(20.) An algebraic *polynomial* consists of two or more monomials connected by the sign + or − ; and such monomials are called the *terms* of the polynomial.

Thus $5a^2+bx$, and $ax^2+3b-5c^2$ are polynomials.

A polynomial composed of *two terms* is called, more definitely, a *binomial*, and one composed of three terms, a *trinomial*.

(21.) The *value* of a polynomial is not affected by changing the *order of its terms*, without changing the *sign* prefixed to any term.

Thus $a+b-c$ is equivalent to $a-c+b$; for the result will evidently be the same, whether b be first added to a, and c then subtracted ; or c be first subtracted, and b afterwards added.

(22.) A polynomial is arranged according to the *powers* of one of its letters, when the *exponents* of that letter increase, or *decrease*, continually in the successive terms.

Thus the polynomial $3a^3+4a^2x-5ab$, is arranged according to the *descending powers* of a, since the exponents of a decrease continually in the successive terms.

The letter,—as a in this example,—according to which the polynomial is arranged, is called the *letter of arrangement*.

(23.) A polynomial is said to be *homogeneous*, when the *sum of the exponents* of the literal factors, is the same in each of its terms.

Thus a^3+2ax^2-bcy is homogeneous, since the sum of the exponents of the literal factors is the same, namely 3, in each term.

Each one of the literal factors composing a term, is called a *dimension* of that term ; and the *degree* of any term is the ordinal of the number of its literal factors or *dimensions*.

Thus $4a^2x$ contains three literal factors, aax, and is therefore of three *dimensions*, or of the *third degree*.

☞ Arrange the polynomial $2x+3a^2x^2-4ax^3+a^3y^2$. according to the *descending powers* of a.—Arrange it according to the *ascending* powers of a.—Which of the terms of this polynomial are *homogeneous*? Tell the number of *dimensions* in each term.—Of what *degree* is each term?

Positive and Negative Quantities.

(24.) A *positive* quantity is one which enters *additively*, and a *negative* quantity is one which enters *subtractively*, into a calculation.

A positive quantity is denoted by the sign +, and a negative quantity by the sign —, prefixed to it.

In the polynomial $a+b-c$, the quantity b is *positive*, while c is *negative.*

A *quantity with neither + nor — prefixed to it*, is understood to be *positive.*

Thus in the preceding polynomial, the first term a is understood to be $+a$; for it may be regarded as $0+a$.

(25.) A *negative* quantity has an effect *contrary to that of an equal positive* quantity, in the expression, or calculation, into which it enters.

This is evident with regard to the *addition* and *subtraction* of the same quantity; the effect of *subtracting* it is contrary to that of adding it.

The effect of *multiplying* by a *negative* quantity is contrary to that of multiplying by an equal *positive* quantity.

For example; $3a\times 2$, $3a$ multiplied by *positive* 2, denotes that $3a$ is to be repeated *additively*, and is equivalent to $3a+3a$.

But $3a\times -2$, $3a$ multiplied by *negative* 2, denotes that $3a$ is to be repeated *subtractively*, and is equivalent to $-3a-3a$.

The same thing is true with respect to *dividing* by a negative, and an equal positive quantity.

The positive and negative signs are sometimes used to distinguish quantities as estimated in *contrary directions* from a given point or line.

Thus if Motion in one direction be denoted by the *sign* +, then motion in the contrary direction will be denoted by the *sign* —. If North Latitude be considered as *positive*, South Latitude will be *negative.*

CHAPTER II.

ADDITION.—SUBTRACTION.—MULTIPLICATION.—DIVISION.

ADDITION.

(26.) Algebraic ADDITION consists in finding the *simplest expression* for the value of two or more quantities connected together by the sign + or —, and this equivalent expression is called the *sum of the quantities.*

The *simplest expression* for the value of $5a+3a$, 5 times a plus 3 times a, is $8a$; then $8a$ is the *sum* of $5a$ and $3a$.

The simplest expression for the value of $5a-3a$, 5 times a minus 3 times a, is $2a$; then $2a$ is the sum of $5a$ and $-3a$.

In the second example, observe that *adding* $-3a$ is equivalent to *subtracting* $3a$; the Adding of a *negative* quantity being the same as the Subtracting of an equal *positive* quantity.

Addition of Monomials.

(27.) *Similar* terms with *like signs*, are added together by taking the sum of their *coefficients*, annexing the common *literal factor*, and prefixing the common *sign.*

Thus $3ax+2ax$ is $5ax$; just as 3 *cents* + 2 *cents* is 5 *cents.*

And $-3a-2a$ is $-5a$; for $3a$ to be subtracted and $2a$ to be subtracted make $5a$ to be subtracted.

☞ What is the Sum of $4a$ and $3a$? Of ax and $5ax$? Of $-$[illegible] and $-7x$? Of $5a^2$, $2a^2$, and $4a^2$? Of $-2b$, $-3b$, and $-5b$?

(28.) Two *equal similar* terms with contrary signs, when added together, mutually *cancel each other*; that is, their sum is 0.

Thus $3a-3a$ is 0; that is, 0 is the simplest expression for the value of $3a-3a$, and is therefore the *sum* of the two terms.

So $5a+x-5a$ is equal to x; for $5a$ and $-5a$ *cancel each other*, and x is therefore the sum of the three given terms.

☞ What is the Sum of $2ax$ and $-2ax$, that is, the simplest expression for the value of $2ax-2ax$? What is the Sum of $7b$ and $-7b$? Of a, $3x$, and $-a$? Of $3y^2$, $-3y^2$, and $5b^2$? Of ax^2, 5, and -5? Of $-abc$, $+a^2x$, and abc?

(29.) Two *unequal similar* terms with contrary signs, are added together by taking the *difference of their coefficients*, annexing the common literal factor, and prefixing the *sign of the greater term.*

Thus $7a-3a$ is $4a$; just as 7 *cents* – 3 *cents* is 4 *cents*; that is, the sum of $7a$ and $-3a$ is $4a$.

And $3a-7a$ is $-4a$. For $-7a$ is equal to $-3a-4a$ (27); then $3a-7a$ is equal to $3a-3a-4a$; $3a$ and $-3a$ cancel each other (28), and leave $-4a$.

☞ What is the Sum of $5ab$ and $-3ab$? Of $3a^2c$ and $-a^2c$? Of $9ax^2$ and $-4ax^2$? Of $-a^2x$ and $3a^2x$? Of $5ac$ and $-5ac$?

What is the simplest expression for the value of $2b-5b$, that is, the Sum of $2b$ and $-5b$? *How do you reason in finding that sum?*

What is the Sum of $3a^2$ and $-8a^2$? and how do you reason in finding it? What is the Sum of ax^2 and $-2ax^2$? and how do you reason in finding it? What is the Sum of $-11a^2$ and $5a^2$? and how do you reason in finding it?

(30.) The Sum of two or more *dissimilar* terms can only be indicated by arranging them in a Polynomial, so that each term shall have its *given sign prefixed to it.*

Thus the Sum of ax and bc^2 can only be indicated as $ax+bc^2$; and the Sum of ax and $-bc^2$ as $ax-bc^2$.

What is the Sum of $3a$ and $5b$? Of $5a$ and $-b$? Of $3a^2$ and $2x^2$? Of ax and by? Of $-3b$ and $-4y$?

What is the Sum of $3b$, $5b$, and $6c$? Of a^2, bx, and $3a^2$? Of $4c$ and $-3y$? Of $-ab$, $3xy$, and $-2c$? Of $3a^2$, $4b$, and $2c$? Of $5a$, -5, and $5b$?

The preceding principles. and the following Rule, provide for the cases in Algebraic Addition.

RULE I.

(31) *For the Addition of Algebraic Quantities.*

1. Find the *positive* and the *negative* Sum of similar terms, separately, (27), and then add together the *similar sums*, (28) and (29).

2. Connect the results thus found, and the *dissimilar* terms, in a Polynomial, prefixing to each term its proper sign. (30).

EXAMPLE.

To add together $4a^2+2bc-xy$, $2a^2-3bc+5y$, $bc-a^2+3xy$, and $cy-2a^2-5bc$.

$$\begin{array}{l}
4a^2+2bc-xy \\
2a^2-3bc+5y \\
-a^2+bc+3xy \\
-2a^2-5bc+cy \\
\hline
3a^2-5bc+2xy+cy+5y
\end{array}$$

The sum of the positive terms $2a^2$ and $4a^2$ is $6a^2$, and the sum of the similar *negative* terms $-2a^2$ and $-a^2$ is $-3a^2$, (27). Adding together these two similar sums, $6a^2$ and $-3a^2$, *the result is* $3a^2$, (29).

The sum of the positive terms bc and $2bc$ is $3bc$, and the sum of the similar *negative* terms $-5bc$ and $-3bc$ is $-8bc$. Adding together these two similar sums, $3bc$ and $-8bc$, *the result is* $-5bc$.

The sum of the similar terms $3xy$ and $-xy$ is $2xy$.

The results thus found, namely, $3a^2$, $-5bc$, and $2xy$, and the *dissimilar* terms cy and $5y$, are connected in a Polynomial.

EXERCISES.

1. Add together $ab+3c^2-2x$, $3ab-c^2+5x$, $5c^2-2ab+y$, and $4ab-c^2+x$. *Ans.* $6ab+6c^2+4x+y$.
2. Add together $4a^2-6b+3$, $3a^2+6b-7$, $5b+a^2-bc$, and $a^2-bc+10$. *Ans.* $9a^2+5b+6-2bc$.
3. Add together $2b^2+bc+x$, $3b^2-2bc+y$, $bc-3b^2+5x$, and $b^2+bc+3y$. *Ans.* $3b^2+bc+6x+4y$.
4. Add together $5a-6b^3+3$, $2b^3-4a-7$, $7a-b^3+c$, and $a+3b^2+2c+4$. *Ans.* $9a-5b^3+3c+3b^2$.
5. Add together $a^2b-5c+xy$, $4a^2b+8c-xy$, $a^2b+c+3xy$, and $a^2+3c-13$. *Ans.* $6a^2b+7c+3xy+a^2-13$.

6. Add together a^3+2a^2-3bc, $2a^3-a^2+bc$, $-a^3-a^2+b^2$, and $3a^3+3a^2+3b^2$. *Ans.* $5a^3+3a^2-2bc+4b^2$.

7. Add together $2x+ab^2-3y^2$, $2ab^2-3x+y^2$, $-ab^2+5x-y^2$, and $5x+ab^2-y$. *Ans.* $9x+3ab^2-3y^2-y$.

8. Add together $4a-3b^2+cx^2$, b^2-3a+3, $2a+3b^2-cx^2$, and $2b^2-2a-9$. *Ans.* $a+3b^2-6$.

9. Add together $7b-2a^2-xy$, $5a^2-6b+3xy$, $b-3a^2+4$, and $a^2-b-3xy$. *Ans.* $b+a^2-xy+4$.

10. Add together $3b^2-2a^3+13$, $3a^3-2b^2-5$, $4ab+7a^3-3$, and $2b^2-a^3+ab$. *Ans.* $3b^2+7a^3+5+5ab$.

11. Add together ax^2-2y+b, $2y+2ax^2-3b$, $4ax^2-y-b$, and $2b-3ax^2+b$. *Ans.* $4ax^2-y$.

12. Add together $2c^2+a^2+3bc$, $5c^2-3a^2-2bc$, c^2+2a^2-bc, and $b+bc-3$. *Ans.* $8c^2+bc+b-3$.

13. Add together $ab+a^2c-5$, $3ab-3a^2c+7$, $2a^2c-2ab-3$, and $ab+a^2c+5$. *Ans.* $3ab+a^2c+4$.

14. Add together $3b^2-2a^2x+b$, $-b^2+3a^2x-3b$, b^2-a^2x+c, and $3b^2+b-3c$. *Ans.* $6b^2-b-2c$.

15. Add together $2a^2+3-ac$, $3a^2-7+ac$, $3ac-5a^2+9$, and $3ac+4-a^2$. *Ans* $-a^2+9+6ac$.

16. Add together b^2c+2-y^2, y^2-3b^2c-10, $2b^2c-3+2y^2$, and b^2-y^2+5. *Ans.* $-6+y^2+b^2$.

17. Add together $a^3+bc-\frac{1}{2}c$, $2a^3-bc+\frac{3}{4}c$, $3a^3+3bc-\frac{1}{4}c$, and a^3+bc+c. *Ans.* $7a^3+4bc+c$.

18. Add together $b^3-3a^2+\frac{2}{3}c^2$, $2b^2+5a^2+\frac{1}{2}c^2$, $b^2-\frac{1}{3}c+2$, and $5b^2+3a^2-2c+3y$. *Ans.* $b^3+5a^2+1\frac{1}{6}c^2+8b^2-2\frac{1}{3}c+2+3y$.

(32.) When the same Polynomial is enclosed in two or more *parentheses*; (11), this polynomial enters into a calculation in the same manner as the *common factor* of similar monomials.

Thus the Sum of $3(a+b)$ and $5(a+b)$, is evidently $8(a+b)$.

19. Add together $2(a-3b)+c$, $3(a-3b)-3c$, $-4(a-3b)+c$, and $5(a-3b)+3c$. *Ans.* $6(a-3b)+2c$.

20. Add together $3a^2+2(a-c^2)$, $a^2-3(a-c^2)$, $4a^2+5(a-c^2)$ and $-7a^2+(a-c^2)$. *Ans.* $a^2+5(a-c^2)$.

21. Add together $5(a+b-c)+3b^2$, $3(a+b-c)-2b^2$, $2(a+b-c)$ and $3b^2+2y$, $+b^2$. *Ans.* $10(a+b-c)+5b^2+2y$.

22. Add together $2ab^2+3a(b+c^2)$, $ab^2-2a(b+c^2)$, $a(b+c^2)$, and $-ab^2-3a(b+c^2)$. *Ans.* $2ab^2-a(b+c^2)$.

23. Add together $5+\frac{1}{2}(c-d+m)+2b$, $1+\frac{3}{4}(c-d+m)-b+2$, and $2(c-d+m)+\frac{1}{2}b$. *Ans.* $8+3\frac{1}{4}(c-d+m)+1\frac{1}{2}b$.

SUBTRACTION.

(33.) Algebraic SUBTRACTION consists in finding the *difference* between the two quantities; that is, in finding *what quantity added to the quantity subtracted*, will produce that from which the subtraction is made.

Thus the difference found by subtracting $3a$ from $5a$ is $2a$, because $2a$ added to $3a$ produces $5a$, (27).

But the difference found by subtracting $-3a$ from $5a$ is $8a$; because $8a$ must be added to $-3a$ to produce $5a$, (29).

In the second example, observe that *subtracting* $-3a$ has an effect contrary to that of subtracting $3a$ in the first example, (25); the Subtracting of a *negative* quantity being the same as the Adding of an equal *positive* quantity.

Subtraction of Monomials.

(34.) A *monomial is subtracted* from another quantity, by changing the *sign* of the monomial, and then *adding it*, algebraically, to the other quantity.

Thus $5a$ subtracted from $2a$ gives $-3a$, because $-3a$ must be added to $5a$, to produce $2a$, (29); and this difference, $-3a$, will be found by changing $5a$ to $-5a$, and adding the $-5a$ to the $2.a$

☞ What difference will be found by subtracting $5a$ from $9a$? and why is that difference *true?* By subtracting $9a$ from $5a$?

What difference will be found by subtracting $3ab$ from $4ab$? and why is that difference true? By subtracting $-5x$ from $-10x$? and why is that difference true? By subtracting $7ax$ from $-3ax$.

(35.) When the two quantities are *dissimilar*, a monomial is subtracted by changing its *sign*, and then connecting it with the other quantity, in a Polynomial.

Thus $2x$ subtracted from $3a$ gives $3a-2x$; and $-2x$ subtracted from $3a$ gives $3a+2x$, (34), (30).

☞ What difference will be found by subtracting $2a$ from $3x$? By subtracting $-3b$ from $5ax$? By subtracting ax from $10b$? By subtracting $4y$ from $-9x^2$?

From the preceding we derive

RULE II.

(36.) *For the Subtraction of Algebraic Quantities.*

Change the sign of each term *to be subtracted*, + to — and — to +, or conceive these signs to be changed, and then proceed as in Algebraic Addition.

EXAMPLE.

To subtract $a^2+2ab-3x-z$ from $3a^2-5ab+5x-y$.

$$\begin{array}{r} 3a^2-5ab+5x-y \\ a^2+2ab-3x-z \\ \hline 2a^2-7ab+8x-y+z \end{array}$$

Having set the polynomial which is to be subtracted under the one from which the subtraction is to be made, we conceive each term in the second line to have its *sign changed;* then $-a^2$ added to $3a^2$ makes $2a^2$; $-2ab$ added to $-5ab$ makes $-7ab$; and $3x$ added to $5x$ makes $8x$. These results, together with the $-y$ and $+z$, are connected in a polynomial.

EXERCISES.

1 From $4ab+3c^2-xy$, subtract $2ab-c^2+3xy-2z$.
Ans. $2ab+4c^2-4xy+2z$.

2 From $5a^2+3bc-cd+3x$, subtract $2a^2-bc+3cd$.
Ans. $3a^2+4bc-4cd+3x$.

3 From $7a^2b-abc+4xy$, subtract $a^2b+5abc-xy+mz$.
Ans. $6a^2b-6abc+5xy-mz$.

4 From $8a^3-4a^2b-2bc+10$, subtract $3a^3+a^2b-5$.
Ans. $5a^3-5a^2b-2bc+15$.

5. From $3b^2c+abx+2xy^2$, subtract $b^2c-3abx+2xy^2-3$.
Ans. $2b^2c+4abx+3$.

6. From $4a^2c-3c^2b+6y+20$, subtract $3c^2b+6y+3by$.
Ans. $4a^2c-6c^2b+20-3by$.

7. From $-3a+5b^2-7xy$, subtract $3b^2+2a-xy+bc$.
Ans. $-5a+2b^2-6xy-bc$.

8. From $2ab+3bc-2ax^2+15$, subtract $7ab+3bc-10$.
Ans. $-5ab-2ax^2+25$.

9. From $5ax-3bc+2b^2$, subtract $6ax-bc-3b^2+y^2$.
Ans. $-ax-2bc+5b^2-y^2$.

10. From $a^2b+3abc+b^2c-7$, subtract $\frac{1}{2}a^2b-abc+\frac{1}{2}b^2c$.
Ans. $\frac{1}{2}a^2b+4abc+\frac{1}{2}b^2c-7$

When one quantity is to be subtracted from the Sum of two or more quantities, the operation will be expedited by changing the signs of the *subtractive* quantity, and then *adding all the quantities together.*

11. From the Sum of $2a+3bc-1$, and $3a-4bc+3$, subtract $4a-bc-2+y$. *Ans.* $a+4-y$.

12. From the Sum of $ax^2-6b^2+3ay^2$, and $2ax^2+6b^2-ay^2$, subtract $-3ax^2+b^2+2ay^2-5$. *Ans.* $6ax^2-b^2+5$.

13. From the Sum of $3ab+bc-5x^2$, and $ab-3bc+x^2-3$, subtract $5bc-3x^2-4+y^2$. *Ans.* $4ab-7bc-x2+1-y^2$.

14. From the Sum of $5x-3xy+7y^2$, and $5xy-3x-4y^2-9$, subtract $-x+xy-3y^2+7$. *Ans.* $3x+xy+6y^2-16$.

15. From the Sum of $2a-b+3cd$, $5a+3b-cd-3$, and $a+2b-cd$, subtract $4a-5b-cd$. *Ans.* $4a+9b+2cd-3$.

16. From the Sum of $3x^2+y-5xy$, $2x^2-3y+xy$, and $4y-3x^2-xy+5$, subtract $x^2+5y-xy-3$. *Ans.* $x^2-3y-4xy+8$.

17. From the Sum of $2ay^2-2b+3ax$, $2ay^2+b-ax$, and $3b-ay^2+3-y$, subtract $3ay^2+4b-ax+y$.
Ans. $-2b+3ax+3-2y$.

18. From the Sum of $3a^2+xy^2-2by$, $5a^2+3xy^2-3by$, and $3xy^2+4a^2+by$, subtract $2a^2-xy^2-5by+5$.
Ans. $10a^2+8xy^2+by-5$.

19. From the Sum of $a^2b^2-3y^2+5xy$, $3a^2b^2+3y^2-2xy$, and $5y^2+3a^2b^2-xy+6$, subtract $a^2b^2+xy-y^2-3$.
Ans. $6a^2b^2+6y^2+xy+9$.

20. From the Sum of $5y^2-3ax-2bc$, $4ax-2y^2+5bc$, and $3ax+y^2-bc+5$, subtract $ax-bc+3y^2-1$.
Ans. $y^2+3ax+3bc+6$.

21. From the Sum of a^2y-x^2+bc-d, $3x^2-3bc+5d$, and $a^2y+3x^2-4bc-d$, subtract $5x^2-2bc+7d$.
Ans. $2a^2y-4bc-4d$.

22. From the Sum of $5b^2+3ax+2y$, $3b^2-ax-y+1$, and $4ax-b^2-5y+2$, subtract $3b^2+2ax-y+10$.
Ans. $4b^2+4ax-3y-7$.

23. From the Sum of $7a+6x^2-y^2+1$, x^2+3y^2-c-3, and $7x^2+y^2-3c+5$, subtract $5a+2x^2-2y^2+5c+2$.
Ans. $2a+12x^2+5y^2-9c+1$.

24. From the Sum of $2\ (a-x)+y$, $3(a-x)+2y$, and $5(a-x)-y$, subtract $4(a-x)+y-2$, (32). *Ans.* $6(a-x)+y+2$.

25. From the Sum of $2b(a+c)^2+3$, $b(a+c)^2-1$, and $3b(a+c)^2-2$, subtract the sum of $b(a+c)^2+b$, and $5b(a+c)^2-b$.
Ans. 0.

Indicated Subtraction.

(37.) It is sometimes expedient merely to *indicate* the subtraction of a quantity, without performing the operation.

To denote the subtraction of a positive monomial, nothing more is necessary than to place the *sign* — before it; thus $a-b$ denotes that b, that is, *positive* b, is to be subtracted from a.

The subtraction of a *negative* monomial, will be denoted by enclosing the quantity, with its negative sign, in a (   ), and prefixing the sign — to the parenthesis.

Thus $a-(-b)$ denotes that *negative* b is to be subtracted from a. When the subtraction is performed, the expression becomes $a+b$.

The subtraction of a *polynomial* will be denoted by enclosing the polynomial in a (   ), and prefixing the sign — to the parenthesis.

Thus $a-(b+c+d)$; in which the *sign* — denotes that the *enclosed polynomial* is to be subtracted. When the subtraction is performed, the expression becomes $a-b-c-d$.

Change of Signs in a Polynomial.

(38.) The *signs of the value* of a Polynomial, will evidently be changed, by changing the signs of all the terms of the Polynomial.

The value of a Polynomial is therefore not affected by *changing the signs* of any or all of its terms, enclosing those terms in a (   ), and prefixing the *sign* — to the parenthesis.

Thus $a+b-c$ is equivalent to $a-(-b+c)$, or $a-(c-b)$; for if $(c-b)$ be subtracted, as is required by the *sign* — prefixed to it, the result will be $a+b-c$.

26. Under what different forms may the value of the polynomial $ab+2cd-3x+5$ be expressed?

27. Under what different forms may the value of the polynomial $ax-3y+4b^2-5c-7$ be expressed?

MULTIPLICATION.

(39.) Algebraic MULTIPLICATION consists in finding the Product of one quantity taken as many times, *additively*, or *subtractively*, as there are units in another quantity.

The quantity to be multiplied is called the *multiplicand*, and the multiplying quantity the *multiplier:* both together are called the *factors* of the product, (12).

When the Multiplier is *positive*, the multiplicand is repeated positively, or is repeatedly *added.*

Thus $5a \times 3$, $5a$ multiplied by *positive* 3, is $5a+5a+5a$, equal to $15a$; the multiplicand $5a$ being repeatedly *added.*

When the Multiplier is *negative*, the multiplicand is repeated *negatively*, or is repeatedly *subtracted.*

Thus $5a \times -3$, $5a$ multiplied by *negative* 3, is $-5a-5a-5a$, which is equal to $-15a$; the multiplicand $5a$ being repeatedly *subtracted*

Multiplication of Monomials.

(40.) The Product of two *monomials* consists of the product of their *coefficients* multiplied into all their *literal factors.*

For example, $3ac \times 2x$ is equal to $6acx$; for the factors may be taken in the order $3 \times 2acx$, which becomes $6acx$ by substituting 6 for 3×2.

☞ What is the Product of $3a \times 4b$? and how do you reason in finding that product? What is the Product of $5a^2b \times 2x$? and how do you reason in finding it? Of $7ac^2 \times xy$? Of $xy^2 \times 5$? Of $3 \times 7a^2b^2$?

(41.) When the *same letter* occurs in both the monomials multiplied together, its *exponent* in the Product will be the *sum* of its exponents in the two factors.

Thus $3a^2x \times 2ax$, or $3a^2x^1 \times 2a^1x^1$, is equal to $6a^2axx$, (40), and this product becomes $6a^3x^2$ by substituting a^3 and x^2 for their equivalents a^2a and xx, (13) and (16).

Observe that the exponents of a and x in the product $6a^3x^2$, are the *sums* of the exponents of the same letters, respectively, in the two monomials which are multiplied together.

☞ What is the Product of $4a^3\times2a^2$? Of $3ax\times5ax^2$? Of $7a\times2a^3b$? Of $4ac\times acb$? Of $a^2b^3\times5b$? Of $8a^2cb\times c$? Of $a^2b\times a^3b^2$?

Sign of the Product.

(42.) The Product of two quantities is *positive* when they both have the *same sign*, and *negative* when they have contrary signs;—in other words, *like signs* produce $+$, and *unlike signs* produce $-$, in multiplying.

When both the quantities are *positive*, their product is evidently *positive*,—as in common Arithmetic. Thus $3a\times2$ is $3a+3a$, equal to $6a$.

When both the quantities are *negative*, their product is *positive*, because such product results from repeatedly *subtracting a negative quantity*, (36).

For example, in $-3a\times-2$ the multiplier -2 denotes that $-3a$ is to be taken twice *subtractively*; and since the *sign* of a quantity is changed in subtracting it, the product is $3a+3a$, or $6a$.

When one of the quantities is *positive* and the other *negative*, their product is *negative* because it results from repeatedly *subtracting* a *positive*, or adding a *negative* quantity.

Thus in $3a\times-2$, the $3a$ is to be taken twice subtractively, and the product is therefore $-3a-3a$, equal to $-6a$.

☞ What is the Product of $-2x\times-3$? and how do you reason on the *sign* of the product? Of $-3ab\times-2$? Of $-5a^2\times-3$? Of $-x\times-5$?

What is the Product of $5ax\times-4$? and how do you reason on the *sign* of the product? Of $4a^2c\times-3$? Of $7x^2\times-5$? Of $b^2\times-9$.

What is the Product of $-2ab\times3$? and how do you reason on the *sign* of the product? Of $-3a^2x\times4$? Of $-3a^2\times2$? Of $-c^3\times5\frac{1}{2}$?

What is the Product of $4x^2\times-7$? and how do you reason on the *sign* of the product? Of $5ay\times-3$? Of $-3a\times-4a$? Of $6b\times-5$?

When the Multiplier or the Multiplicand is 0.

(43.) When *either* of the two factors multiplied together is 0, the product will be 0; for it is evident that 0 repeated any number of times, produces only 0; and the product is the same when the multiplicand and the multiplier are taken *the one for the other.*

Thus 5×0 is 0; $a\times 0$ is 0; $(a+b-c)\times 0$ is 0.

The preceding principles and the two following rules, embrace the subject of Algebraic Multiplication.

RULE III.

(44.) *To Multiply a Monomial into a Polynomial.*

Find the product of the monomial into *each term* of the polynomial. separately, and connect these products in a polynomial with the proper *sign prefixed to each,* (40), (41), (42).

EXAMPLE.

To multiply $3ab+bc^2-2xy-5$ by $2a^2x$

$$\begin{array}{l} 3ab+bc^2-2xy-5 \\ 2a^2x \\ \hline 6a^3bx+2a^2bc^2x-4a^2x^2y-10a^2x \end{array}$$

It will be most convenient to multiply from *left to right.* The first term of the multiplicand is *positive,* and the multiplier being also *positive,* the first term, $6a^3bx$, of the product is *positive.* In like manner the second term, $2a^2bc^2$, of the product is positive, &c.

EXERCISES.

1. Multiply $2a^2-3b+c-2y$ by 3. *Ans.* $6a^2-9b+3c-6y$.
2. Multiply $3x^2-xy-y^2+4$ by 2. *Ans.* $6x^2-2xy-2y^2+8$.
3. Multiply $4b+c^2-3y+1$ by $-y$. *Ans.* $-4by-c^2y+3y^2-y$.
4. Multiply $-a+3b-2x^2-3$ by $-2x$. *Ans.* $2ax-6bx+4x^3+6x$.
5. Multiply ab^2-b^3+2c-y by -5. *Ans.* $-5ab^2+5b^3-10c+5y$.
6. Multiply $-3+ax-1+by$ by $-a$. *Ans.* $3a-a^2x+a-aby$.
7. Multiply $7x+by^2-4c+d^2$ by 4. *Ans.* $28x+4by^2-16c+4d^2$.
8 Multiply a^2b-c^2+x-2 by $3a$. *Ans.* $3a^3b-3ac^2+3ax-6a$.

RULE IV.

(45.) *To Multiply a Polynomial into a Polynomial.*

Multiply one of the polynomials by *each term*, separately, of the *other polynomial*, and add together the several products.

EXAMPLE.

To multiply $2a^2+4ac-c^2$ by $3a-5c$

$$\begin{array}{l}
2a^2+4ac-c^2 \\
3a\ \ -5c \\
\hline
6a^3+12a^2c-3ac^2 \\
\qquad -10a^2c-20ac^2+5c^3 \\
\hline
6a^3+\ \ 2a^2c-23ac^2+5c^3.
\end{array}$$

We multiply the first polynomial by $3a$, and then by $-5c$, according to the preceding Rule. The two products thus obtained are set with similar terms one under another, and added together.

The correctness of this Rule, as well as of the preceding one, will be evident upon considering, *first*, that if *each part* of a quantity be multiplied, the *whole* will be multiplied; and, *secondly*, that if one quantity be multiplied by *each part of another quantity*, the former will be multiplied by the *whole of the latter*.

EXERCISES.

9. Multiply a^2-2a+5 by $a+3$. *Ans.* a^3+a^2-a+15.
10. Multiply $2x^2+3x-1$ by $x-5$. *Ans.* $2x^3-7x^2-16x+5$.
11. Multiply $a^2-2ax+x^2$ by $a+x$. *Ans.* $a^3-a^2x-ax^2+x^3$.
12. Multiply $x^2+3xy+y^2$ by $x-y$. *Ans.* $x^3+2x^2y-2xy^2-y^3$.
13. Multiply $b^2-3bc-c^2$ by $2b+c$. *Ans.* $2b^3-5b^2c-5bc^2-c^3$.
14. Multiply $2a^2+3ac-c^2$ by $a-2c$. *Ans.* $2a^3-a^2c-7ac^2+2c^3$.
15 Multiply $2b^2-2bx+x^2$ by $2b-x$. *Ans.* $4b^3-6b^2x+4bx^2-x^3$.
16 Multiply b^3+b^2+b by b^3+b^2. *Ans.* $b^6+2b^5+2b^4+b^3$.
17 Multiply x^2-2c^2+2y by c^2+y. *Ans.* $c^2x^2-2c^4+x^2y+2y^2$.
18. Multiply a^2+bx+y by a^2-bx. *Ans.* $a^4+a^2y-b^2x^2-bxy$.

19 Find the Product of $(2a^2-4ab+2b^2)(3a-2b)$.
Ans. $6a^3-16a^2b+14ab^2-4b^3$.

20 Find the Product of $(3x^2-2xy+2y^2)(2x^2+3xy)$.
Ans. $6x^4+5x^3y-2x^2y^2+6xy^3$.

21 Find the Product of $(2a^2+3ab-b^2)(a^2-ab+b^2)$.
Ans. $2a^4+a^3b-2a^2b^2+4ab^3-b^4$.

22 Find the Product of $(3x^2-2xy+5)(x^2+2xy-3)$
Ans. $3x^4+4x^3y-4x^2-4x^2y^2+16xy-15$.

23. Find the Product of the three polynomials $(a+b)$, $(a-b)$, and (a^2+ab+b^2) *Ans.* $a^4+a^3b-ab^3-b^4$.

DIVISION.

(46.) Algebraic DIVISION consists in finding a factor or Quotient, which, multiplied into a given *divisor*, will produce a given *dividend*.

Thus $6a^2x\div 2a$, $6a^2x$ divided by $2a$, gives the *quotient* $3ax$, because this factor, multiplied into the divisor $2a$, produces the dividend $6a^2x$.

Division of Monomials.

(47.) The Quotient of two *monomials* will be found by dividing the *coefficient* of the divisor into that of the dividend, and *subtracting the exponents* in the divisor from those of the same letters, respectively, in the dividend.

Thus $10a^3x^2y\div 5a^2x$ gives the quotient $2axy$, because this quotient, *multiplied into the divisor* $5a^2x$, produces the dividend $10a^3x^2y$; and $2axy$ is found by dividing 5 into 10, and diminishing the exponents of a and x in the dividend by the exponents of a and x in the divisor.

The dividend being the *product* of the divisor and quotient, the *exponents* in the dividend are the *sums* of the exponents of the same letters in the divisor and quotient, (41); hence the exponents in the quotient will always be found by subtracting as above.

☞ What is the Quotient of $6a^3x^2y\div 3ax$? and how do you prove that quotient *true?* What is the Quotient of $4ab^2c\div b$? and how do you prove that quotient true? What is the Quotient of $8a^2bx\div 2a$? and how do you prove that quotient true?

When any Exponent is reduced to 0.

(48.) Any quantity whatever with *exponent* 0 is equivalent to *unity;* and a *factor* with this exponent may therefore be canceled.

For example, $a^2 \div a^2$ gives the quotient a^0, by subtracting the exponent of the divisor from that of the dividend.

But any quantity contains itself *once,* and therefore $a^2 \div a^2$ also gives the quotient 1; and these two quotients, a^0 and 1, must be *equivalent to each other.*

Since a in the preceding illustration may represent any quantity we please, any quantity whatever with *exponent* 0, is equivalent to 1, or is a *symbol of unity.*

To divide $10a^3b$ by $5a^3$.

Subtracting the exponent in the divisor from the exponent of a in the dividend, we find the quotient $2a^0b$. The factor a^0 is equivalent to 1, and may therefore be canceled without affecting the value of the quotient.

The quotient of $10a^3b \div 5a^3$ may therefore be given under the two different forms $2a^0b$ and $2b$.

☞ Under what *two different forms* may the Quotient of $6a^2x \div 2a^2$ be represented? Of $8ab^2 \div 2b^2$? Of $10abc \div 5a$? Of $9ax^2 \div x^2$? Of $7abc^2 \div 7c^2$? Of $5ax \div ax$? Of $3aby^2 \div ay^2$? Of $10x^2 \div x^2$?

Sign of the Quotient.

(49.) The Quotient of two quantities is *positive* when they both have the *same sign,* and *negative* when they have contrary signs;—in other words, *like signs* produce +, and *unlike signs* produce —, in dividing.

Thus $+ax \div +a$ gives $+x$, because $+a \times +x$ is $+ax$;
$-ax \div -a$ gives $+x$, because $-a \times +x$ is $-ax$;
$+ax \div -a$ gives $-x$, because $-a \times -x$ is $+ax$;
and $-ax \div +a$ gives $-x$, because $+a \times -x$ is $-ax$; (42.)

From these examples, it will be seen that the *sign of the quotient* must be such that the quotient multiplied into the *divisor* shall produce the *dividend.*

It thus appears that the rule for the sign of the *quotient,* is the same as that for the sign of the *product,* of two quantities. The learner must be careful not to apply this rule in finding the *sum,* or *difference,* of two quantities.

☞ What is the Quotient of $4a^2x \div 2$? and how do you reason on the *sign* of the quotient? Of $6ac^2 \div 2c$? Of $9x^3y \div 3xy$? Of $a^3 \div a$?

What is the Quotient of $-7a^3 \div -a^2$? and how do you reason on the *sign* of the quotient? Of $-12c^2 \div -4$? Of $-a^3b \div -ab$? Of $-b^4 \div b^2$?

What is the Quotient of $8a^2 \div -a^2$? and how do you reason on the *sign* of the quotient? Of $10ab \div -2b$? Of $4a^2c \div -a$? Of $5x^2 \div -x^2$?

What is the Quotient of $-7a^3 \div 7a$? and how do you reason on the *sign* of the quotient? Of $-12ac \div 3a$? Of $-9a^3b \div 3$? Of $-20a^4 \div 4$?

What is the Quotient of $10a^3 \div 5a$? and why is that quotient true? What is the Quotient of $12x^2 \div -3$? and why is that quotient true? What is the Quotient of $-9a^3by \div 3a^3$? and why is that quotient true? What is the Quotient of $-20a^4x^3 \div -5ax$? and why is that quotient true? What is the Quotient of $-100axy \div -xy$? and why is that quotient true?

When the Dividend or the Divisor is 0.

(50.) The quotient of 0 divided by any quantity, is 0; but the quotient of any quantity divided by 0, is *infinitely great.*

First, let a represent any quantity we please; then
$0 \div a$ gives the quotient 0, because this quotient multiplied into the *divisor* a, produces the *dividend* 0, (43).

Secondly, a divisor may be taken so small as to be contained a *great number of times* in any given dividend, represented by a; if the divisor be still diminished, the quotient will be increased; and if the divisor be diminished *without limit,* the quotient will be *increased without limit.*

If therefore the divisor were dimished to 0, the quotient would be *unlimited,* that is, *infinitely great.*

The character ∞ is the sign of *infinity;*
we have then $0 \div a$ equal to 0; and $a \div 0$ equal to ∞, *infinity.*

The preceding principles, and the two following Rules, embrace the subject of Algebraic Division, when the Quotient is an *integral* quantity.

RULE V.

(51.) *To Divide a Monomial into a Polynomial.*

Find the quotient of the monomial divided into each term of the polynomial, separately, and connect these quotients in a polynomial, with the proper *sign prefixed to each.* (47), (48), (49).

EXAMPLE.

To divide $20a^2x-15ax^2+30axy^2-5ax$ by $5ax$.

$$\begin{array}{r|l} 5ax) & 20a^2x-15ax^2+30axy^2-5ax \\ \hline & 4a \quad - \quad 3x \quad + \quad 6y^2 \quad -1 \end{array}$$

Dividing into the first term of the dividend, we find the ***quotient term*** $4a$; dividing into the second term of the dividend, we find the quotient $-3x$; and so on, through the dividend.

EXERCISES.

1. Divide $3x^3+6x^2+3ax-15x$ by $3x$. *Ans.* $x^2+2x+a-5$.
2. Divide $2ab-6a^2x+8a^3y-2a$ by $2a$. *Ans.* $b-3ax+4a^2y-1$
3. Divide $14a^2-7ab+21ax-21a$ by $7a$. *Ans.* $2a-b+3x-3$.
4. Divide $a^3x+3a^2x^2-6ax^3+3ax$ by a. *Ans.* $a^2x+3ax^2-6x^3+3x$.
5. Divide $5b^2-10b^3+5b^4y-15b^5$ by $5b$. *Ans.* $b-2b^2+b^3y-3b^4$.
6. Divide $bx^2+2x^3-8cx^4+7x^5$ by x^2. *Ans.* $b+2x-8cx^2+7x^3$.
7. Divide $4a^4-8a^3-4a^2b+8a$ by $2a$. *Ans.* $2a^3-4a^2-2ab+4$.
8. Divide $ay^4+a^2y^3-a^3y^2-ay$ by ay. *Ans.* $y^3+ay^2-a^2y-1$.
9. Divide $-y+by^2-5y^3+3y^4$ by $-y$. *Ans.* $1-by+5y^2-3y^3$.
10. Divide $3b^3-9b^2+12b-15$ by $+3$. *Ans.* b^3-3b^2+4b-5.
11. Divide $-c^5+3c^4-6c^3+c^2$ by $-c^2$. *Ans.* c^3-3c^2+6c-1.
12. Divide $8y^3-4ay+a^2y^2-3y$ by y. *Ans.* $8y^2-4a+a^2y-3$
13. Divide $-10+20a-15a^2+20$ by -5. *Ans.* $2-4a+3a^2-4$.
14. Divide $a^3b-a^2b^2+a^3b^2-a^4b$ by ab. *Ans.* $a^2-ab+a^2b-a^3$.
15. Divide $x^4y^4+x^3y^3-x^2y^2-xy$ by xy. *Ans.* $x^3y^3+x^2y^2-xy-1$.

RULE VI.

(52.) *To Divide a Polynomial into a Polynomial.*

1. Arrange the divisor and dividend according to the *ascending* or the descending *powers* of the same letter, (22).

2. Divide the *first term* of the divisor into the *first term* of the dividend; multiply the whole divisor by the *quotient* term; subtract the product from the dividend; divide into the remainder, as before, and so on,—observing to connect the several quotient terms in a polynomial, with the proper *sign* prefixed to each.

EXAMPLE.

To divide $6a^4x^2+3a^3x^3-4a^2x^4+x^6$ by $2ax+x^2$

$$
\begin{array}{lrrrl}
2ax+x^2\,) & 6a^4x^2+3a^3x^3 & -4a^2x^4+x^6 & & (\,3a^3x-2ax^3+x^4 \\
 & 6a^4x^2+3a^3x^3 & & & \\
\hline
 & & -4a^2x^4+x^6 & & \\
 & & -4a^2x^4-2ax^5 & & \\
\hline
 & & & 2ax^5+x^6 & \\
 & & & 2ax^5+x^6 & \\
\hline
\end{array}
$$

The divisor and dividend are here arranged according to the *descending* powers of a, or the *ascending* powers of x.

The first term $2ax$ of the divisor, divided into the first term $6a^4x^2$ of the dividend, gives the quotient term $3a^3x$. Multiplying the whole divisor by this quotient term, and subtracting the product from the dividend, the remainder is $-4a^2x^4+x^6$.

We next divide the first term of the divisor into the first term, $-4a^2x^4$, of the remainder, and find the quotient term $-2ax^3$; &c.

By this Rule the divisor is multiplied by each *part of the quotient*, and the successive products subtracted from the dividend. When the dividend is thus *exhausted*, the product of the divisor and quotient is *equal to the dividend;* and the quotient is thus proved to be correct.

The Divisor, when a polynomial, is sometimes set on the left of the dividend, and the quotient under the divisor. This arrangement is thought to afford greater facility in multiplying the divisor by the quotient term.

EXERCISES.

16. Divide $a^2-2ax+x^2$ by $a-x$. *Ans.* $a-x$.
17. Divide $x^3-3ax^2+3a^2x-a^3$ by $x-a$. *Ans.* $x^2-2ax+a^2$.
18. Divide $6a^4+9a^2-15a$ by $3a^2-3a$. *Ans.* $2a^2+2a+5$.
19. Divide $a^4+a^2x^2+x^4$ by a^2-ax+x^2. *Ans.* a^2+ax+x^2.
20. Divide $2x^3-19x^2+26x-16$ by $x-8$. *Ans.* $2x^2-3x+2$.
21. Divide $4x^4-5x^2c^2+c^4$ by $2x^2-3xc+c^2$. *Ans.* $2x^2+3xc+c^2$.
22. Divide $a^4-2a^2x^2+x^4$ by $a^2+2ax+x^2$. *Ans.* $a^2-2ax+x^2$.
23. Divide $12-4a-3a^2+a^3$ by $4-a^2$. *Ans.* $3-a$.
24. Divide $4y^4-9y^2+6y-1$ by $2y^2+3y-1$. *Ans.* $2y^2-3y+1$.
25. Divide $2x^4-32$ by $x-2$. *Ans.* $2x^3+4x^2+8x+16$.
26. Divide $4a^5-64a$ by $2a-4$. *Ans.* $2a^4+4a^3+8a^2+16a$.
27. Divide $6y^6-96y^2$ by $3y-6$. *Ans.* $2y^5+4y^4+8y^3+16y^2$.
28. Divide a^4+4x^4 by $a^2-2ax+2x^2$. *Ans.* $a^2+2ax+2x^2$.
29. Divide a^5-x^5 by $a^4+a^3x+a^2x^2+ax^3+x^4$. *Ans.* $a-x$.
30. Divide $y^4+4y^2z^2-32z^4$ by $y+2z$. *Ans.* $y^3-2y^2z+8yz^2-16z^3$.

(53.) The *indicated Product* of two or more factors is divided by any quantity, when *either of the factors* is divided by that quantity.

Thus to divide $3a^2x(x^2+y^2)$ by $3x$, we divide the factor $3a^2x$, and find the quotient $a^2(x^2+y^2)$.

In the following exercises, the first of the given *factors* may be divided by the given divisor.

31. Divide $(a^2+2ay+y^2)(b^3-cd^2+3)$ by $a+y$.
Ans. $(a+y)(b^3-cd^2+3)$.
32. Divide $(2x^3-6ax^2+6a^2x-2a^3)(c^2+3cy-y^2)$ by $x-a$.
Ans. $(2x^2-4ax+2a^2)(c^2+3cy-y^2)$.
33. Divide $(a^4+4y^4)(3xy^2-5y^3+3y^4-4)$ by $a^2-2ay+2y^2$.
Ans. $(a^2+2ay+2y^2)(3xy^2-5y^3+3y^4-4.)$
34. Divide $(8a^4-2a^3x-13a^2x^2-3ax^3)(y^2+2xy)(y^3+3x^2y^2-x^3)$ by $4a^2+5ax+x^2$. *Ans.* $(2a^2-3ax)(y^2+2xy)(y^3+3x^2y^2-x^3)$.

CHAPTER III.

COMPOSITE QUANTITIES.—COMMON MEASURE.—COMMON MULTIPLE.

Composite and Prime Quantities.

(54.) A *composite* quantity is one which is the *product* of two factors, each differing from *unity;* and a *prime* quantity is one which is not the product of such factors.

Thus $3a^2$ is a *composite,* while $a+b$ is a *prime* quantity.

☞ Is ab a composite or a prime quantity? Is $a+5$ a composite or a prime quantity? a^2? $7x^2$? $2a+3b$? $5a-5c^2$?

Decomposition of Quantities.

(55.) *Decomposing* a composite quantity consists in resolving it into its *factors.* Any *divisor* of the quantity, and the corresponding *quotient,* are two factors into which it may be resolved, (46.)

Thus if the binomial $3x+6ax$ be divided by $3x$, the quotient will be $1+2a$; the binomial may therefore be resolved into the factors

$3x(1+2a)$, $3x$ into the binomial $1+2a$.

Resolve $2a+4ax-6a^2x^2$ into component factors.
Resolve $a^2x-3ax^2+8ay^2$ into component factors.
Resolve $4a^2+a^2b-5a^2y$ into component factors.
Resolve $2a^3-3ax+7a^4y$ into component factors.
Resolve $5ax+5a^2x-10a^3x^3$ into component factors.
Resolve $7ab-14ab^2-14abx$ into component factors.

A *monomial factor* may generally be found by mere inspection, and a composite polynomial be resolved into a monomial and a polynomial factor, as above.

No general Rule can be given for resolving a polynomial into the *polynomial factors* of which it may be composed. But there are particular Binomials which have well known *binomial divisors,* by means of which such Binomials may be resolved into two factors, (55).

The following divisions will be found, on trial, to terminate without remainders.

(56.) The Difference of two quantities will divide the difference of any *powers of the same degree* of those quantities.

Thus $a-b$ will divide a^2-b^2, or a^3-b^3, or a^4-b^4, &c.

(57.) The Sum of two quantities will divide the *sum* of any *odd powers* or the difference of any *even* powers, of the same degree, of those quantities.

Thus $a+b$ will divide a^3+b^3, or a^5+b^5, or a^7+b^7, &c.,
also $a+b$ will divide a^2-b^2, or a^4-b^4, or a^6-b^6, &c.;

The factors of certain Binomials and Trinomials may also be known from the manner in which the *products* and *squares* of binomials are formed. This will be seen in the following propositions.

(58.) The Product of the *sum* and *difference* of two quantities is equal to the difference of the *squares* of those quantities.

Thus $(a+b)(a-b)$ is equal to a^2-b^2; and this last binomial may therefore be divided by either $a+b$ or $a-b$.
This proposition, it will be observed, is included in the two preceding ones, (56.) (57.)

☞ What is the Product of $(a+x)(a-x)$? What is the Product of $(a+5)(a-5)$? Of $(3+y)(3-y)$? Of $(x-1)(x+1)$?

(59.) The Square of the *sum* of two quantities is equal to the sum of the *squares* plus *twice* the product of the two quantities.

Thus $(a+b)(a+b)$, that is, the *square* of $a+b$, is equal to
a^2+b^2+2ab or $a^2+2ab+b^2$.
This trinomial may therefore be divided by $a+b$.

☞ What is the Product of $(a+x)(a+x)$, or the *square* of $a+x$? What is the Square of $a+y$? Of $x+3$? Of $a+c$? Of $y+5$?

(60.) The Square of the *difference* of two quantities is equal to the sum of the *squares* minus *twice* the product of the two quantities.

Thus $(a-b)(a-b)$, that is, the square of $a-b$, is equal to
a^2+b^2-2ab or $a^2-2ab+b^2$.
This trinomial may therefore be divided by $a-b$.

☞ What is the Product of $(a-x)(a-x)$, or the square of $a-x$? What is the Square of $a-y$? Of $y-2$? Of $b-x$? Of $1-y$?

The preceding principles will be found applicable to the following

EXERCISES.

1. Resolve a^2-x^2 into component factors.
   Ans. $(a-x)(a+x)$
2. Resolve a^3+y^3 into component factors.
   Ans. $(a+y)(a^2-ay+y^2)$
3. Resolve a^3-x^3 into component factors.
   Ans. $(a-x)(a^2+ax+x^2)$
4. Resolve a^4-y^4 into component factors.
   Ans. $(a^2+y^2)(a+y)(a-y)$
5. Resolve a^3-8x^3 into component factors.
   Ans. $(a-2x)(a^2+2ax+4x^2)$.
6. Resolve $1-8y^3$ into component factors.
   Ans. $(1-2y)(1+2y+4y^2)$
7. Resolve $1+27x^3$ into component factors.
   Ans. $(1+3x)(1-3x+9x^2)$.
8. Resolve $8a^3-27y^3$ into component factors.
   Ans. $(2a-3y)(4a^2+6ay+9y^2)$.
9. Resolve $a^3x^3+c^3y^3$ into component factors.
   Ans. $(ax+cy)(a^2x^2-axcy+c^2y^2)$.
10. Resolve a^4-16x^4 into component factors.
    Ans. $(a^2+4x^2)(a+2x)(a-2x)$.
11. Resolve $a^2+2ax+x^2$ into component factors.
    Ans. $(a+x)(a+x)$.
12. Resolve $a^2-2ay+y^2$ into component factors.
    Ans. $(a-y)(a-y)$.
13. Resolve $a^2+4ax+4x^2$ into component factors.
    Ans. $(a+2x)(a+2x)$.
14. Resolve $9a^2-6ay+y^2$ into component factors.
    Ans. $(3a-y)(3a-y)$.
15. Resolve $4a^2+12ax+9x^2$ into component factors.
    Ans. $(2a+3x)(2a+3x)$
16. Resolve $a^2x^2-2ax+1$ into component factors.
    Ans. $(ax-1)(ax-1)$
17. Resolve $1+4xy+4x^2y^2$ into component factors.
    Ans. $(1+2xy)(1+2xy)$.
18. Resolve $4a^2x^2-12abxy+9b^2y^2$ into component factors.
    Ans. $(2ax-3by)(2ax-3by)$
19. Resolve $9a^4x^4+24a^2cx^2y+16c^2y^2$ into component factors.
    Ans. $(3a^2x^2+4cy)(3a^2x^2+4cy)$.
20. Resolve $16a^4x^2-32a^2cxy^2+16c^2y^4$ into component factors.
    Ans. $(4a^2x-4cy^2)(4a^2x-4cy^2)$.

(61.) A Trinomial may be resolved into two *binomial* factors, whenever one of its three terms is a *square*, another the *sum of the products* of the square root of that term multiplied into any two quantities, and the remaining term the *product of the same two quantities.*

For example, let it be required to decompose the trinomial

$$a^2-a-12.$$

The first term is the *square* of a; the second term is the *sum of the products* of a multiplied into 3 and -4; and the third term is the product of 3 and -4.

Now, from the manner in which the product of two binomials is formed, it is evident that

$$a^2-a-12 \text{ is equal to } (a+3)(a-4).$$

In like manner the following Trinomials may be decomposed.

21. Resolve $a^2+7a+12$ into component factors.
Ans. $(a+3)(a+4)$.

22. Resolve x^2+x-30 into component factors.
Ans. $(x+6)(x-5)$.

23. Resolve $a^2-8a-20$ into component factors.
Ans. $(a-10)(a+2)$.

24. Resolve $y^2-10y-39$ into component factors.
Ans. $(y-13)(y+3)$.

25. Resolve $a^2x^2+9ax+18$ into component factors.
Ans. $(ax+6)(ax+3)$

26. Resolve $a^2y^2-11ay+28$ into component factors.
Ans. $(ay-7)(ay-4)$

27. Resolve $3ab^2+21ab+36a$ into component factors.

The given quantity may be resolved first into

$$3a(b^2+7b+12);$$

and the *trinomial factor* thus obtained may be resolved into $(b+3)(b+4)$

Ans. $3a(b+3)(b+4)$.

28. Resolve $5x^2-5x-60$ into component factors.
Ans. $5(x-4)(x+3)$.

29. Resolve $2ab^2+18ab+36a$ into component factors.
Ans. $2a(b+6)(b+3)$.

30. Resolve $4ax^3+10ax^2+6ax$ into component factors.
Ans. $ax(2x+3)(2x+2)$.

COMMON MEASURE.

(62.) One quantity is called a *measure* of another, or is said to *measure* another, if it will divide the other, without a remainder; and

A *common measure* of two or more quantities is any quantity that will *divide each of them*, without a remainder.

Thus $5a$ is a common measure of $10ab$ and $5a^2+15ab^2$.

☞ Name a Common measure of $4a^2b$ and $6ab^2$. Of $9a^2$ and $3a^2+a^2c$. Of $5abc+3a^2c$ and $4ac$. Of a^2x and a^2x+a^3y.

Greatest Common Measure.

(63.) The *greatest common measure* of two or more quantities, is the greatest quantity that will divide each of them, without a remainder.

Thus $3a^2$ is the greatest common measure of $6a^2x+3a^3y$ and $3a^2b-6a^3cd$, since it is the greatest quantity that will divide each of these binomials, without a remainder.

Here it may be remarked, that when we speak of algebraic quantities as being *greater* or *less*, we have reference to the *coefficients* and *exponents* of the same letters, and not to any particular values which the letters may be supposed to represent.

Thus a^2 is, algebraically, always *greater* than a; though its *numerical* value would be less than that of a, if the latter should be taken to represent a *proper fraction*, as $\frac{1}{2}$, $\frac{2}{3}$, &c.

☞ What is the Greatest Common Measure of $3a^2x+4a^3x$ and $a^3y-5a^2y^3$? Of $5a^3c+10a^3c^2$ and $10a^2b-5a^3bx$? Of $2ax^2-6a^2y$ and $4a^2x+8ay^2-2a$?

When two quantities have no common measure but *unity*, they are said to be *prime to each other*.

(64.) The greatest common measure of two or more quantities is composed of all the factors which are *common to those quantities*, that is, all the factors which are found in *each quantity*.

For example, the common factors in $3axy^2$ and $6a^2y^2z$ are 3, a, and y^2; and $3ay^2$ is the greatest common measure of the two quantities.

But as no general Rule can be given for resolving a Polynomial into the *polynomial* factors of which it may be composed, we make the Rule for finding the greatest common measure depend on the following proposition ;—

(65.) The greatest common measure of two or more quantities, is the same as that of the *least* of those quantities and the *remainder*, or remainders, if any, after dividing the least into the other, or each of the others.

We may suppose any two quantities to be represented by

$$a \text{ and } na+b\,;$$

n being some *integral* number, and b less than a.

It is plain that *any measure* of a will also measure na, n times a; and, measuring na, if it measure $na+b$, it must also measure b. Hence there can be no common measure of a and $na+b$ which is not a common measure of a and b; the greatest common measure, therefore, of a and b, will be the greatest common measure of a and $na+b$.

But b is the *remainder* after dividing a into $na+b$; hence the proposition is true for two quantities; and in like manner it may be proved for three or more quantities.

RULE VII.

(66.) *To find the Greatest Common Measure of two Quantities.*

1. Divide one of the quantities into the other, and the *remainder* into the divisor, and so on, until there is no remainder: the last divisor will be the greatest common measure required.—But,

2. When any factor is contained in *each term of the divisor*, without being contained in each term of the *corresponding dividend*,—such factor must be *canceled*, before dividing.—And

3. When the first term of the divisor is not a *measure* of the first term of the dividend—multiply the several terms of the latter by some quantity which will thus render its first term divisible, without a remainder.

NOTE—This Rule might be readily extended to three or more quantities; but we seldom or never have occasion to find the greatest common measure of more than two algebraic quantities.

EXAMPLE.

To find the greatest common measure of the polynomials

$$4a^2x-5ax^2+x^3 \text{ and } 3a^3-3a^2x+ax^2-x^3.$$

$$\begin{array}{l|l}
\dfrac{4a^2x-5ax^2+x^3}{4a^2-5ax+x^2} & 3a^3-3a^2x+ax^2-x^3 \\
 & 4 \\ \cline{2-2}
 & 12a^3-12a^2x+4ax^2-4x^3\,(3a \\
 & 12a^3-15a^2x+3ax^2 \\ \cline{2-2}
 & \qquad 3a^2x+ax^2-4x^3 \\
 & \qquad 4 \\ \cline{2-2}
 & \qquad 12a^2x+4ax^2-16x^3\,(3x \\
 & \qquad 12a^2x-15ax^2+3x^3 \\ \cline{2-2}
 & \qquad\qquad 19ax^2-19x^3
\end{array}$$

$$\begin{array}{l|l}
\dfrac{19ax^2-19x^3}{a-x} & 4a^2-5ax+x^2\,(4a-x \\
 & 4a^2-4ax \\ \cline{2-2}
 & \quad -ax+x^2 \\
 & \quad -ax+x^2
\end{array}$$

In this example the factor x is contained in each term of the *divisor* $4a^2x-5ax^2+x^3$, without being contained in each term of the dividend; we therefore cancel this factor, and take $4a^2-5ax+x^2$ for a divisor.

Then, the first term $4a^2$ of this divisor not being a *measure* of the first term $3a^3$ of the dividend, we multiply the dividend by 4, which renders the first term $12a^3$ divisible, without a remainder.

We also multiply the *remainder* $3a^2x+ax^2-4x^3$, still regarded as a dividend, by 4, to make the first term divisible, without a remainder.

In the next remainder, $19ax^2-19x^3$, the exponent of a, the *letter of arrangement*, being less than in the first term of the divisor,—we divide this remainder, after canceling the factor $19x^2$, into the former divisor, which now becomes the dividend.

The divisor $a-x$ completes the operation, and is the *greatest common measure* required.

It remains to elucidate the Rule.

The greatest common measure of the two polynomials, is the same as that of the divisor $4a^2x-5ax^2+x^3$ and the *remainder* after the first division, (65). On the same principle the greatest common measure of the first divisor and remainder, is the same as that of the first remainder and the second remainder ; and so on, to the last remainder, which becomes the last divisor.

Hence, the last remainder, or divisor—being the greatest common measure of *itself* and the preceding divisor, and so on to the first remainder and divisor—is the greatest common measure of the given polynomials.

Again ; since the greatest common measure is composed of all the factors found in *each* of the two polynomials, (64), it is not affected by canceling a factor,—as x in the first divisor,—which is found in only one of them ; nor by introducing a *new factor*,—as in multiplying the first dividend by 4,—into only one of them,—since these expedients do not interfere with the original common factors.

The suppression of a factor, as x in this example, which is peculiar to the divisor, is necessary ; for, otherwise, the dividend must be multiplied by this factor to render its first term divisible,—and this would introduce a *new common factor* into the two polynomials, and thus increase the greatest common measure, (64).

In the preceding Example, we might have taken the first remainder, $3a^2x+ax^2-4x^3$, for a divisor, and the first divisor for the dividend. The operation from that point would be as follows ;—

$$\begin{array}{l|l}
\dfrac{3a^2x+ax^2-4x^3}{3a^2+ax-4x^2} & \begin{array}{l} 4a^2-5ax+x^2 \\ 3 \\ \hline 12a^2-15ax+3x^2 \ (4 \\ 12a^2+4ax-16x^2 \\ \hline \quad -19ax+19x^2 \end{array}
\end{array}$$

$$\begin{array}{l|l}
\dfrac{-19ax+19x^2}{-a+x} & \begin{array}{l} 3a^2+ax-4x^2 \ (-3a-4x \\ 3a^2-3ax \\ \hline \quad 4ax-4x^2 \\ \quad 4ax-4x^2 \end{array}
\end{array}$$

The greatest common measure is here found to be $-a+x$, which is the one before determined, with its *signs changed.*

The effect of changing the signs in the divisor, would only be to change the signs in the quotient resulting from the division. Hence

(67.) A Common measure of two or more quantities may have all its *signs changed,* without ceasing to be a common measure of those quantities.

The Rule which has been given, directs that one of the two quantities be divided by the other, &c., without distinguishing the divisor from the dividend. We therefore remark here, that the same common measure will be found by taking *either* of the two quantities for the divisor, and the other for the dividend.

If the divisor and dividend in the preceding Example, be interchanged with each other, the operation will be as follows ;—

$$\begin{array}{r|l}
 & 4a^2x-5ax^2+x^3 \\
 & 3a \\ \cline{2-2}
3a^3-3a^2x+ax^2-x^3 & 12a^3x-15a^2x^2+3ax^3 \quad (4x \\
 & 12a^3x-12a^2x^2+4ax^3-4x^4 \\ \cline{2-2}
 & -3a^2x^2-ax^3+4x^4
\end{array}$$

$$\begin{array}{r|l}
-3a^2x^2-ax^3+4x^4 & \\ \cline{1-1}
-3a^2-ax+4x^2 & 3a^3-3a^2x+ax^2-x^3 \quad (-a \\
 & 3a^3+a^2x-4ax^2 \\ \cline{2-2}
 & -4a^2x+5ax^2-x^3 \\
 & 3 \\ \cline{2-2}
 & -12a^2x+15ax^2-3x^3 \quad (4x \\
 & -12a^2x-4ax^2+16x^3 \\ \cline{2-2}
 & 19ax^2-19x^3
\end{array}$$

$$\begin{array}{r|l}
19ax^2-19x^3 & \\ \cline{1-1}
a-x & -3a^2-ax+4x^2 \quad (-3a+4x \\
 & -3a^2+3ax \\ \cline{2-2}
 & -4ax+4x^2 \\
 & -4ax+4x^2
\end{array}$$

In this case the factor x is contained in each term of the dividend, without being contained in each term of the divisor. This factor, therefore, does not enter into the composition of the greatest common measure, (64), and might be canceled before dividing ; and this would simplify the operation. Hence

(68.) When any factor is common to all the terms of the *dividend*, and not to those of the divisor, such factor may be suppressed, without affecting the greatest common measure of the two quantities

It follows also from proposition (64), that

(69.) When the same factor is contained in all the terms of two polynomials, their greatest common measure may be found by multiplying this factor into the greatest common measure found for the polynomials *without this factor.*

For example, to find the greatest common measure of

$$a^2x-x^3 \text{ and } a^2x+ax^2-2x^3.$$

The factor x is contained in all the terms, and the two polynomials *without this factor* are a^2-x^2 and $a^2+ax-2x^2$.

The greatest common measure of these will be found to be $a-x$; then, multiplying x into $a-x$, we find $ax-x^2$ for the greatest common measure of the given polynomials.

EXERCISES.

1. Find the greatest common measure of

$$a^3-x^3 \text{ and } a^4-x^4.$$

Ans. $a-x$.

2. Find the greatest common measure of

$$a^2-ax-2x^2 \text{ and } a^2-3ax+2x^2.$$

Ans. $a-2x$.

3. Find the greatest common measure of

$$3a^2-2a-1 \text{ and } 4a^3-2a^2-3a+1.$$

Ans. $a-1$.

4. Find the greatest common measure of

$$x^2+2ax+a^2 \text{ and } x^3-a^2x.$$

Ans. $x+a$.

5. Find the greatest common measure of

$$2x^3-16x-6 \text{ and } 3x^3-24x-9.$$

Ans. x^3-8x-3.

6. Find the greatest common measure of

$$a^2-5ax+4x^2 \text{ and } a^3-a^2x+3ax^2-3x^3.$$

Ans. $a-x$.

7. Find the greatest common measure of

$$a^2-2ax+x^2 \text{ and } a^3-a^2x-ax^2+x^3.$$

Ans. $a^2-2ax+x^2$.

8. Find the greatest common measure of

$$6a^2+7ax-3x^2 \text{ and } 6a^2+11ax+3x^2.$$

Ans. $2a+3x$.

9. Find the greatest common measure of

$$7x^2-23xy+6y^2 \text{ and } 5x^3-18x^2y+11xy^2-6y^3.$$

Ans. $x-3y$.

10. Find the greatest common measure of

$$a^4+a^2y^2+y^4 \text{ and } a^3-2a^2y+2ay^2-y^3.$$

Ans. a^2-ay+y^2.

11. Find the greatest common measure of

$$3a^5+6a^4x+3a^3x^2 \text{ and } a^3x+3a^2x^2+3ax^3+x^4.$$

Ans. $a^2+2ax+x^2$.

12. Find the greatest common measure of

$$x^3-ax^2-8a^2x+6a^3 \text{ and } x^4-3ax^3-8a^2x^2+18a^3x-8a^4.$$

Ans. $x^2+2ax-2a^2$.

COMMON MULTIPLE.

(70.) One quantity is called a *multiple* of another, if it can be *measured* by the other, that is, divided by it, without a remainder; and

A *common multiple* of two or more quantities is any quantity that can be divided by each of them, without a remainder.

Thus $8a^3x^2$ is a common multiple of $2a^2x$ and $4x^2$.

☞ Name a Common Multiple of $3ab$ and $5c^2$. Of ax^2 and $2ay$. Of $3a^2$ and $4x^2$. Of $5bc$ and x^2. Of y^3 and $5x^3$.

Least Common Multiple.

(71.) The *least common multiple* of two or more quantities, is the smallest quantity that can be divided by each of them, without a remainder.

Thus $6ax^2$ is the least common multiple of $6a$ and $6x^2$.

☞ What is the Least Common Multiple of $2ab$ and $3b^2$? Of $4x$ and $4y^2$? Of $5a$ and $3a^2b$? Of ab and $2bc^2$? Of $7ax$ and ax^2? Of $3a^2$ and $6ay^2$? Of axy and $5x^2y^2$?

(72.) The least common multiple of two or more quantities is composed of the *smallest selection of factors* that includes the factors of each given quantity.

For example, take the quantities $3ab^2c$ and $6a^2bxy$,
Resolving these two quantities into their *prime factors*, we have

$$3abbc \text{ and } 3 \times 2\ aab\ xy.$$

If we take $3 \times 2\ aa\ bbc\ xy$, we shall have the smallest selection of factors that includes the factors of each of the two given quantities.

Then the product $6a^2b^2cxy$ is the least common multiple, because it is the smallest quantity that each of the given quantities will divide, without a remainder.

(73.) The least common multiple of *two quantities*, is equal to their *product* divided by their greatest common measure.

For since the greatest common measure of two quantities, is composed of all the factors which are *common to those quantities*, (64), these factors will enter *twice* in the *product* of the quantities.

If, therefore, the product be divided by the greatest common measure, the quotient will contain only those factors which are *common* to the two quantities, and those which are *peculiar to each* of them; and these are the factors of the least common multiple, (72).

RULE VIII.

(74.) *To find the Least Common Multiple of Two or more Quantities.*

1. Set the quantities in a line, from left to right, and divide any *two* or more of them by any *prime* quantity, greater than unity, that will divide them, without a remainder,—placing the quotients and the *undivided* quantities in a line below.

2. Divide any two or more of the quantities in the lower line, as before; and so on, until no two quantities in the lowest line can be so divided. The product of the *divisors* and quantities in the *lowest line*, will be the least common multiple required.

3. If no two of the given quantities can be divided as above, the *product of all the quantities* will be their least common multiple.

EXAMPLE.

To find the least common multiple of

$$3ax^2, \quad 6a^2y, \text{ and } 3y-4y^3$$

$$\begin{array}{rccc}
3) & 3ax^2 & 6a^2y & 3y-4y^3 \\ \hline
a) & ax^2 & 2a^2y & 3y-4y^3 \\ \hline
y) & x^2 & 2ay & 3y-4y^3 \\ \hline
 & x^2 & 2a & 3-4y^2
\end{array}$$

In the first operation we divide $3ax^2$ and $6a^2y$ by 3, and set down $3y-4y^3$ without dividing it; and in like manner in the second operation. In the third operation we set down x^2 without dividing it.

Then $3ayx^2\times 2a\times(3-4y^2)$, equal to $18a^2x^2y-24a^2x^2y^3$, is the least common multiple of the three given quantities.

This Rule depends on proposition (72): the divisors and quantities in the lowest line, are the smallest selection of factors that includes the factors of each given quantity.

EXERCISES.

1. Find the least common multiple of
ax^2, $2a^2y$, $4y+y^2$, and ax^2+4x^2,
Ans. $8a^3x^2y+2a^3x^2y^2+32a^2x^2y+8a^2x^2y^2$.

2. Find the least common multiple of
$2ay^2$, $4ay^2$, $2x-4x^2$, and x^3+ax^2.
Ans. $4ax^3y^2-8ax^4y^2+4a^2x^2y^2-8a^2x^3y^2$.

3 Find the least common multiple of
$3a^2$, ax^2, $3a+6a^2$, and x^3-3x^2.
Ans. $3a^2x^3+6a^3x^3-9a^2x^2-18a^3x^2$.

4. Find the least common multiple of
$4y^3$, $2ay^2$, $5a-5ab$, and $10a-5$.
Ans. $40a^2y^3-40a^2by^3-20ay^3+20aby^3$.

5. Find the least common multiple of
$5a$, $10ab$, $3y+3y^2$, and $6y^3+3y^2$.
Ans. $90aby^3+60aby^4+30aby^2$.

6. Find the least common multiple of

$4,\ 4x^2,\ 8y-8,$ and $2x^3-ax^3$.

Ans. $16x^3y-16x^3-8ax^3y+8ax^3$

7. Find the least common multiple of

$10,\ 5ax,\ 4a-8y^2,$ and $2x+6x^2$.

Ans. $20a^2x-40axy^2+60a^2x^2-120ax^2y^2$

8. Find the least common multiple of

$3y^2,\ 12y,\ 5x^2-10,$ and $4y-8y^3$.

Ans. $60x^2y^2-120y^2-120x^2y^4+240y^4$

9 Find the least common multiple of

$14,\ 7a,\ 2y^2-4,$ and $28+7ay$.

Ans. $56ay^2+112a+14a^2y^3-28a^2y$.

---

10. Find the least common multiple of

a^4-x^4, and $a^3-a^2x-ax^2+x^3$, (73).

Ans. $a^5-a^4x-ax^4+x^5$.

11. Find the least common multiple of

$x^2+2bx+b^2$ and x^3-b^2x. (73).

Ans. $x^4+bx^3-b^2x^2-b^3x$.

12. Find the least common multiple of

$a^2-3ab+2b^2$ and $a^2-ab-2b^2$. (73).

Ans. $a^3-2a^2b-ab^2+2b^3$.

13. Find the least common multiple of

$4a^3-2a^2-3a+1$ and $3a^2-2a-1$. (73).

Ans. $12a^4-2a^3-11a^2+1$.

CHAPTER IV.

FRACTIONS.

(75.) An algebraic FRACTION represents the Quotient of its *numerator* divided by its *denominator.*

Thus $\frac{a}{3x}$ represents the Quotient of a divided by $3x$.

In reading an algebraic Fraction, it will often be necessary to use the terms *numerator* and *denominator*, to avoid ambiguity in reference to the division which is expressed.

Thus if the Fraction

$$\frac{a+x}{y}$$

be read, $a+x$ divided by y, it might be understood that only x is to be divided by y. But the true sense would be conveyed by saying, *numerator* $a+x$, *denominator* y.

A Fraction is thus employed to represent the Quotient, when the divisor is not a factor of the dividend. The quotient in this case may also be represented by means of

Negative Exponents.

(76.) Any quantity with a *negative exponent* is equivalent to a *unit* divided by the same quantity with *the sign of its exponent changed.*

Thus a^{-2}, a with exponent -2, is equivalent to $\frac{1}{a^2}$.

For a^2a^{-2} is equal to a^0, (41);
and by *dividing each of these equals* by a^2,

we find a^{-2} equal to $\frac{a^0}{a^2}$ or $\frac{1}{a^2}$, (48).

☞ What is the fractional *equivalent* of a^{-3}? and how is it proved? Of x^{-4}? and how is it proved? Of y^{-5}? and how is it proved? Of $(a+x)^{-2}$? Of $(x-y)^{-3}$?

(77.) When the Divisor is not a factor of the dividend, the Quotient may be represented by a *fraction* (75), or by the dividend *multiplied* into the divisor with the *sign of its exponent changed.*

$a \div x^2$ will give the quotient ax^{-2}, because this quotient multiplied into the *divisor* x^2, produces ax^0, (41), which is equal to a, the dividend, (48).

We have, therefore, ax^{-2} for an *integral*, and $\frac{a}{x^2}$ for a *fractional* form of the quotient of $a \div x^2$.

☞ What is the *integral* form of the Quotient of $a \div b^3$? and how is it proved? Of $x \div y^4$? and how is it proved? Of $1 \div a^2$? and how is it proved? Of $a^2 \div x$? Of $b \div ac^2$? Of $a \div 5$?

Transfer of Factors.

(78.) Any factor may be transferred from the *denominator* to the numerator, and *vice versa*, by changing the *sign of its exponent.*

For example, if we divide a by x^2y, fractionally,

$$\text{we have } a \div x^2y \text{ equal to } \frac{a}{x^2y}.$$

If we divide a by the factors x^2 and y, separately, we shall find

$$a \div x^2 \text{ equal to } ax^{-2}, \text{ (77), and } ax^{-2} \div y \text{ equal to } \frac{ax^{-2}}{y}.$$

Hence a divided by x^2y gives

$$\frac{a}{x^2y} \text{ or } \frac{ax^{-2}}{y},$$

These two quotients being necessarily equal to each other, we see that x^2 may be transferred from the denominator to the numerator, by changing the sign of its exponent.

If we also transfer the factor y, we shall have

$$\frac{a}{x^2y} \text{ equal to } \frac{ax^{-2}y^{-1}}{1} \text{ or } ax^{-2}y^{-1}, \text{ (77).}$$

If we transfer the factor a from the numerator a, or $1a$, to the denominator, we find

$$\frac{a}{x^2y} \text{ equal to } \frac{1}{a^{-1}x^2y}.$$

☞ Under what *different* forms may the Quotient of $a \div bx^2$ be represented? Of $3a \div x^3$? Of $2c \div 3a^4$? Of $ab \div$ [illegible]$^2y^2$?

Reciprocals of Quantities.

(79.) The *reciprocal* of a quantity is a *unit divided* by the quantity,

Thus the reciprocal of a^3 is $\frac{1}{a^3}$ or a^{-3} (76).

(80.) The reciprocal of a Fraction is equivalent to the fraction *inverted;* that is, with its numerator and denominator taken *the one for the other.*

Thus the reciprocal of $\frac{a}{x^2}$, or ax^{-2} (77), is $\frac{1}{ax^{-2}}$, which becomes $\frac{x^2}{a}$ by transferring x^{-2} from the denominator to the numerator, (78).

☞ Under what *different forms* may the Reciprocal of $\frac{a^2}{x}$ be expressed? The reciprocal of $\frac{a}{3c^2}$? The reciprocal of $\frac{ab}{y^3}$? Of $\frac{ax}{cy^2}$?

Constant Value of a Fraction.

(81.) The value of a Fraction *remains the same* when its numerator and denominator are both *multiplied*, or both divided, by the same quantity.

For example, if we multiply both terms of $\frac{a}{x^2}$ by y,

we shall have $\frac{a}{x^2}$ equal to $\frac{ay}{x^2y}$.

For the *first* of these fractions is equal to ax^{-2};
and the second is equal to $ax^{-2}yy^{-1}$, (77).

But yy^{-1} is equal to y^0, (41), and by cancelling this factor, (48), we find the *second* fraction also equal to ax^{-2}. Hence the two fractions are equal to each other.

☞ Prove that $\frac{a^2}{b}$ is equal to $\frac{a^2x}{bx}$. That $\frac{c^2}{x^3}$ is equal to $\frac{c^2m^2}{x^3m^2}$.

Signs of Fractions.

(82.) Since a Fraction represents the *quotient* of its numerator divided by its denominator, a Fraction is *positive* when its numerator and denominator have the same sign, and *negative* when they have contrary signs, (49).

☞ Say whether the Fraction $\frac{a}{b}$ is *positive* or negative. Say whether $\frac{a}{-b}$ is positive or negative. Say whether $\frac{-a}{-b}$ is positive or negative. Say whether $\frac{-ax}{by}$ is positive or negative.

(83.) The sign + or − *prefixed to a Fraction*,—not the sign of either the numerator or denominator,—shows whether the fraction enters *additively* or *subtractively* into a calculation.

Thus $+\frac{-a}{b}$ denotes that the fraction $\frac{-a}{b}$, which is in itself *negative*, (82), is to be *added*, in the calculation into which it enters.

When no sign is prefixed to a fraction, + is always understood.

(84.) The signs of both the numerator and denominator *may be changed*, or the sign of either of them with the *sign prefixed to the Fraction*,—without affecting the value of the fraction.

Thus $\frac{a}{b}$ is equivalent to $\frac{-a}{-b}$, since both these fractions are *positive*, (82).

Also $\frac{a}{b}$ is equivalent $-\frac{-a}{b}$. For the fraction $\frac{-a}{b}$ is *negative*, its numerator and denominator having contrary signs; this negative fraction becomes *positive*, when subtracted, as required by the sign − *prefixed to it*, and it is then equivalent to the first fraction.

A Polynomial is changed from *positive* to negative, or from negative to positive, by changing the sign of *each of its terms*, (38).

For example, if $a-b$ is positive, a must be *greater* than b; then, changing the signs, $b-a$ will be negative.

The Fraction $\frac{a-b}{x-y}$ is therefore equivalent to $\frac{b-a}{y-x}$.

☞ What other changes may be made in the *signs*, without affecting the value, of this fraction?

FRACTIONS REDUCED TO THEIR LOWEST TERMS.

(85.) A Fraction is reduced to *lower terms* by dividing its numerator and denominator by the same common *measure.* This simplifies the fraction, without altering its value, (81).

A *monomial* common measure may usually be known by inspection. Thus to reduce the Fraction

$$\frac{4a^2b}{6ac-8a^2x}.$$

It is obvious that we have only to divide its numerator and denominator by $2a$. This gives us the equivalent Fraction,

$$\frac{2ab}{3c-4ax}.$$

A *binomial* common measure may often be discovered from the principles which have been established for the decomposition of Polynomials.

Thus to reduce the Fraction

$$\frac{a^2-b^2}{a^2+2ab+b^2}.$$

By proposition (57) we can divide the numerator by $a+b$; and by (59), we can divide the denominator by the same quantity

Thus dividing we find the equivalent Fraction

$$\frac{a-b}{a+b}.$$

In all cases in which a Fraction admits of being reduced, we may apply

RULE IX.

(86.) *To Reduce a Fraction to its Lowest Terms.*

Divide the numerator and denominator by their greatest common measure, (66): the quotients will be the lowest terms of the given fraction.

EXERCISES.

1. Reduce $\frac{3ax^2}{3a^2b-9a^3}$ to its lowest terms. *Ans.* $\frac{x^2}{ab-3a^2}$

2. Reduce $\frac{2a^2y+4ay^2}{8a}$ to its lowest terms. *Ans.* $\frac{ay+2y^2}{4}$.

3. Reduce $\frac{a^2-b^2}{a^3+b^3}$ to its lowest terms. (57).

Ans. $\frac{a-b}{a^2-ab+b^2}$.

4. Reduce $\frac{a^4-x^4}{a^5-a^3x^2}$ to its lowest terms.

Ans. $\frac{a^2+x^2}{a^3}$

5. Reduce $\frac{x^2-a^2}{x^2+2ax+a^2}$ to its lowest terms.

Ans. $\frac{x-a}{x+a}$.

6. Reduce $\frac{a^3-ay^2}{a^2+2ay+y^2}$ to its lowest terms.

Ans. $\frac{a^2-ay}{a+y}$.

7. Reduce $\frac{a^2-x^2}{a^3-3a^2x+3ax^2-x^3}$ to its lowest terms.

Ans. $\frac{a+x}{a^2-2ax+x^2}$.

8. Reduce $\frac{2ax^2-a^2x-a^3}{2x^2+3ax+a^2}$ to its lowest terms.

Ans. $\frac{ax-a^2}{x+a}$.

9. Reduce $\frac{a^3+2a^2x+3a^2x^2}{2a^4-3a^3x-5a^2x^2}$ to its lowest terms.

Ans. $\frac{a+2x+3x^2}{2a^2-3ax-5x^2}$.

10. Reduce $\frac{6a^2+7ax-3x^2}{6a^2+11ax+3x^2}$ to its lowest terms.

Ans. $\frac{3a-x}{3a+x}$.

11. Reduce $\frac{a^4+a^2x^2+x^4}{a^4+a^3x-ax^3-x^4}$ to its lowest terms.

Ans. $\frac{a^2-ax+x^2}{a^2-x^2}$.

12. Reduce $\frac{5a^5+10a^4y+5a^3y^2}{a^3y+2a^2y^2+2ay^3+y^4}$ to its lowest terms.

Ans. $\frac{5a^4+5a^3y}{a^2y+ay^2+y^3}$.

FRACTIONS REDUCED TO A COMMON DENOMINATOR.

(87.) Two or more Fractions are said to have a *common denominator*, when they have the same quantity for a denominator.

Thus $\frac{ax}{a+b}$ and $\frac{a-x}{a+b}$ have a common denominator.

Two or more Fractions may often be reduced, very readily, to a common denominator, by multiplying both the numerator and denominator of one or more of them, so as to make the denominator the *same for each.*

For example, to reduce to a common denominator the Fractions

$$\frac{a}{2a-2x} \text{ and } \frac{bc}{3a-3x},$$

We have only to multiply the terms of the first fraction by 3, and those of the second by 2. This gives the equivalent Fractions

$$\frac{3a}{6a-6x} \text{ and } \frac{2bc}{6a-6x}. \quad (81).$$

Observe that this reduction does not alter the values of the given Fractions.

When this method cannot be obviously applied, we adopt

RULE X.

(88.) *To Reduce two or more Fractions to a Common Denominator.*

1. Multiply each numerator by all the denominators *except its own*, for new *numerators;* and multiply all the denominators together, for a *common denominator.*

2. *If the Least Common Denominator* be required,—Find the least common *multiple* of the given denominators, for the Common denominator. Divide this Multiple by the denominator of each given Fraction, and multiply the quotient by the numerator, for the new numerators.

EXAMPLES.

1. To reduce $\frac{a}{3x}$, $\frac{b}{6}$, and $\frac{c}{y+2}$ to a common denominator.

For the new numerators, we have

$$\begin{aligned} &a\,.\,6\,.\,(y+2), \text{ equal to } 6ay+12a; \\ &b\,.\,3x\,.\,(y+2), \quad \text{"} \quad 3bxy+6bx; \\ &\text{and } c\,.\,3x\,.\,6 \quad \text{"} \quad 18cx. \end{aligned}$$

And the common denominator is

$$3x\,.\,6\,.\,(y+2), \text{ equal to } 18xy+36x.$$

The given Fractions are thus reduced to

$$\frac{6ay+12a}{18xy+36x},\ \frac{3bxy+6bx}{18xy+36x},\ \frac{18cx}{18xy+36x},\ \text{respectively.}$$

2. To reduce the same Fractions to the *least common denominator.*

The least common *multiple* of the denominators $3x$, 6, and $y+2$ will be found to be $6xy+12x$, (74), which is the required *denominator.*

Dividing this Multiple by each given denominator, and multiplying the quotients by the given numerators, respectively, we find the new numerators,

$$2ay+4a,\ bxy+2bx, \text{ and } 6cx.$$

The Fractions reduced to their least common denominator, are then

$$\frac{2ay+4a}{6xy+12x},\ \frac{bxy+2bx}{6xy+12x},\ \frac{6cx}{6xy+12x},\ \text{respectively.}$$

Each of the given Fractions should be in its *lowest terms*, before proceeding to find their least common denominator; otherwise, the denominator found will not, in all cases, be the smallest by means of which the values of the several Fractions may be expressed.

In finding a Common Denominator as above, the numerator and denominator of each given fraction are multiplied by the *same quantity.*

Thus in the first example, $\frac{a}{3x}$ has both its terms multiplied by 6 and $y+2$,—producing the new terms $6ay+12a$ and $18xy+36x$.

Hence the values of the given fractions are not altered in reducing them to a common denominator, (81).

EXERCISES.

1. Reduce $\frac{a}{3}$, $\frac{b}{2x}$, and $\frac{c}{x-4}$ to a common denominator.

Ans. $\frac{2ax^2-8ax}{6x^2-24x}$, $\frac{3bx-12b}{6x^2-24x}$, $\frac{6cx}{6x^2-24x}$

2. Reduce $\frac{2}{a^2}$, $\frac{a}{y^2}$, and $\frac{a+c}{1-y}$ to a common denominator.

Ans. $\frac{2y^2-2y^3}{a^2y^2-a^2y^3}$, $\frac{a^3-a^3y}{a^2y^2-a^2y^3}$, $\frac{a^3y^2+a^2cy^2}{a^2y^2-a^2y^3}$.

3. Reduce $\frac{a^2}{x}$, $\frac{c}{3y}$, and $\frac{a+5}{y^2}$ to a common denominator.

Ans. $\frac{3a^2y^3}{3xy^3}$, $\frac{cxy^2}{3xy^3}$, $\frac{3axy+15xy}{3xy^3}$.

4. Reduce $\frac{ax}{2}$, $\frac{a}{3x}$, and $\frac{b-1}{1+b}$ to a common denominator.

Ans. $\frac{3ax^2+3abx^2}{6x+6bx}$, $\frac{2a+2ab}{6x+6bx}$, $\frac{6bx-6x}{6x+6bx}$.

5. Reduce $\frac{a^3}{y}$, $\frac{b}{2x^2}$, and $\frac{a+b}{xy}$ to a common denominator.

Ans. $\frac{2a^3x^3y}{2x^3y^2}$, $\frac{bxy^2}{2x^3y^2}$, $\frac{2ax^2y+2bx^2y}{2x^3y^2}$.

6. Reduce $\frac{a}{2x^2}$, $\frac{b}{y}$, and $\frac{ab}{x^2+x^3}$ to the least common denominator.

Ans. $\frac{ay+axy}{2x^2y+2x^3y}$, $\frac{2bx^2+2bx^3}{2x^2y+2x^3y}$, $\frac{2aby}{2x^2y+2x^3y}$.

7 Reduce $\frac{x^2}{4}$, $\frac{a}{3y}$, and $\frac{a-x}{6y^2}$ to the least common denominator.

Ans. $\frac{3x^2y^2}{12y^2}$, $\frac{4ay}{12y^2}$, $\frac{2a-2x}{12y^2}$.

8. Reduce $\frac{a}{y^2}$, $\frac{b}{5}$, and $\frac{c}{y^3-y^2}$ to the least common denominator.

Ans. $\frac{5ay-5a}{5y^3-5y^2}$, $\frac{by^3-by^2}{5y^3-5y^2}$, $\frac{5c}{5y^3-5y^2}$.

9. Reduce $\frac{y}{4}$, $\frac{a}{2x^2}$, and $\frac{a^2}{2+x}$ to the least common denominator.

Ans. $\frac{2x^2y+x^3y}{8x^2+4x^3}$, $\frac{4a+2ax}{8x^2+4x^3}$, $\frac{4a^2x^2}{8x^2+4x^3}$.

10. Reduce $\frac{x-1}{3}$, $\frac{a^2}{3y}$, and $\frac{c}{6y-3y^2}$ to the least common denominator.

Ans. $\frac{2xy-xy^2-2y+y^2}{6y-3y^2}$, $\frac{2a^2-a^2y}{6y-3y^2}$, $\frac{c}{6y-3y^2}$.

INTEGRAL AND MIXED QUANTITIES REDUCED TO IMPROPER FRACTIONS.

(89.) An *integral* quantity is one which does not contain any *fractional* expression; as $3ax^2$, or $2ab-5xy$.

(90.) A *mixed* quantity is partly *integral* and partly *fractional*; as $3ax^2+\frac{a}{5c}$, or $2ab-\frac{x}{1+y}$.

(91.) An *improper* Fraction is a fraction whose value may be expressed by an integral or a mixed quantity.

Thus $\frac{6a^2+5}{3a}$ is an improper fraction whose value is $2a+\frac{5}{3a}$.

RULE XI.

(92.) *To Reduce an Integral or a Mixed Quantity to an Improper Fraction.*

1. Under an *integral* quantity, regarded as a numerator, set 1 for a *denominator*. Or multiply the integral quantity by any proposed denominator; the product will be the numerator.

2. In a *mixed* quantity, multiply the integral part by the denominator annexed; *add the numerator* to the product when the sign before the fractional part is +, but *subtract the numerator* when this sign is —; and place the result over said denominator.

EXAMPLES.

1. To reduce $3a^2$ to a Fraction whose denominator shall be $a-2x$.

$3a^2$ is the same as $\frac{3a^2}{1}$; and by multiplying both terms of this fraction by the proposed denominator, we have

$$3a^2 \text{ equal to } \frac{3a^3-6a^2x}{a-2x}, \ (81).$$

2. To reduce $3a^2-\frac{a-x^2}{2}$ to an improper Fraction

Multiplying $3a^2$ by the denominator 2, and *subtracting* the numerator $a-x^2$ from the product,—observing to *change the signs* of the numerator, (36),—we have

$$3a^2-\frac{a-x^2}{2} \text{ equal to } \frac{6a^2-a+x^2}{2}.$$

The reason of this operation will be evident, if we consider that, by multiplying the $3a^2$ by 2, we reduce the integral part to a *common denominator* with the fraction annexed to it, according to the first part of the Rule. The operation then consists in subtracting numerator from numerator; and placing their difference over the common denominator.

Another view to be taken of the preceding operations, is, that the integral term $3a^2$ is *multiplied*, and the product is then taken *to be divided*, by the same quantity, namely, the denominator. The value therefore remains the same.

EXERCISES.

1. Reduce $4ax^2$ to a Fraction whose denominator shall be $y+2$.

Ans. $\dfrac{4ax^2y+8ax^2}{y+2}$.

2. Reduce a^2+3x to a Fraction whose denominator shall be $2y^2$.

Ans. $\dfrac{2a^2y^2+6xy^2}{2y^2}$.

3. Reduce $5xy$ to a Fraction whose denominator shall be $3+x^2$.

Ans. $\dfrac{15xy+5x^3y}{3+x^2}$.

4. Reduce y^2-5 to a Fraction whose denominator shall be $3ax^2$.

Ans. $\dfrac{3ax^2y^2-15ax^2}{3ax^2}$.

5. Reduce $a+b$ to a Fraction whose denominator shall be $a-b$.

Ans. $\dfrac{a^2-b^2}{a-b}$.

6. Reduce $5ay^2$ to a Fraction whose denominator shall be a^2-b^2.

Ans. $\dfrac{5a^3y^2-5ab^2y^2}{a^2-b^2}$.

7. Reduce x^2+1 to a Fraction whose denominator shall be $3a^2y$.

Ans. $\dfrac{3a^2x^2y+3a^2y}{3a^2y}$,

8. Reduce ab^2-x to a Fraction whose denominator shall be $1-y$.

Ans. $\dfrac{ab^2-x-ab^2y+xy}{1-y}$

9. Reduce $a^2+x+\frac{a}{y}$ to an improper Fraction.

Ans. $\frac{a^2y+xy+a}{y}$.

10. Reduce $ax^2+\frac{b}{y+1}$ to an improper Fraction.

Ans. $\frac{ax^2y+ax^2+b}{y+1}$

11. Reduce $5a+3b-\frac{a+x}{3}$ to an improper Fraction.

Ans. $\frac{14a+9b-x}{3}$.

12. Reduce $2a-4c+\frac{a-y^2}{5}$ to an improper Fraction.

Ans. $\frac{11a-20c-y^2}{5}$

13. Reduce $2+y^2-\frac{1-y^2}{4}$ to an improper Fraction.

Ans. $\frac{7+5y^2}{4}$.

14. Reduce $a^2+x^2-\frac{a^3-x^3}{a-x}$ to an improper Fraction.

Ans. $\frac{ax^2-a^2x}{a-x}$.

15. Reduce $a-x+\frac{a^2-ax}{x}$ to an improper Fraction.

Ans. $\frac{a^2-x^2}{x}$.

16. Reduce $1+2x-\frac{4x-4}{5x}$ to an improper Fraction.

Ans. $\frac{x+10x^2+4}{5x}$.

17. Reduce $2x+2y+\frac{5y^2}{2x-3y}$ to an improper Fraction.

Ans. $\frac{4x^2-2xy-y^2}{2x-3y}$.

18. Reduce $a+x-\frac{a^3-x^3}{a^2-ax+x^2}$ to an improper Fraction.

Ans. $\frac{2x^3}{a^2-ax+x^2}$.

IMPROPER FRACTIONS REDUCED TO INTEGRAL OR MIXED QUANTITIES.

By reversing the preceding Rule, we have

RULE XII.

(93.) *To Reduce an Improper Fraction to an Integral or a Mixed Quantity.*

Divide the numerator by the denominator for the *integral* part, and set the denominator under the remainder, if any, for the *fractional* part, of the result. Connect the fractional to the integral part by the sign +; or *change the sign* of the numerator or denominator, and connect it by the sign —.

EXAMPLE.

To reduce to an integral or a mixed quantity the Fraction

$$\frac{9a^2c+3ab-2b+y}{3a}$$

Dividing the numerator by the denominator, we find the integral quotient to be $3ac+b$, and the *remainder* $-2b+y$.

Setting the denominater under this remainder, and connecting the fraction so formed to the integral quotient, by the sign +, the result is

$$3ac+b+\frac{-2b+y}{3a}.$$

Or, changing the signs in the numerator $-2b+y$, and connecting the fraction by the sign —, the result, under a somewhat simpler form,

$$\text{is } 3ac+b-\frac{2b-y}{3a}.$$

This form is simpler than the preceding, as it dispenses with one sign in the numerator of the fractional part.

The reason of the preceding operation is evident from the consideration that every Fraction is equal to its numerator divided by its denominator, (75). After obtaining the quotient $3ac+b$,—the divisor $3a$ not being contained in the remainder $-2b+y$, the division of these terms is indicated by setting the divisor under them.

The fractional part of the result must evidently be *added* to the integral part; and this addition is indicated by placing the sign + before the fraction. But the value of the fraction annexed, will not be affected by changing the sign + before it to —, if at the same time we change the signs in the numerator, (84).

The Fraction formed of the divisor and remainder, will be in its *lowest terms*, or not, according as the improper fraction reduced, is, or is not, in its lowest terms.

For, if the dividend and divisor have any common *measure*, the divisor and remainder will have the same common measure, (65).

EXERCISES.

1. Reduce $\frac{a^3-x^3}{a-x}$ to an integral or a mixed quantity.

Ans. a^2+ax+x^2.

2. Reduce $\frac{10x^2-5x+3}{5x}$ to an integral or a mixed quantity.

Ans. $2x-1+\frac{3}{5x}$.

3. Reduce $\frac{a^4-y^4}{a-y}$ to an integral or a mixed quantity.

Ans. $a^3+a^2y+ay^2+y^3$.

4. Reduce $\frac{x^4-3x^2y^2+ax}{x^2-3y^2}$ to an integral or a mixed quantity.

Ans. $x^2+\frac{ax}{x^2-3y^2}$.

5. Reduce $\frac{4a^2x^2-3ay-2b}{2a^2}$ to an integral or a mixed quantity.

Ans. $2x^2-\frac{3ay+2b}{2a^2}$.

6. Reduce $\frac{9ax^2-2x+3}{3a}$ to an integral or a mixed quantity.

Ans. $3x^2-\frac{2x-3}{3a}$.

7. Reduce $\frac{a^3+y^3}{a+y}$ to an integral or a mixed quantity.

Ans. a^2-ay+y^2.

8. Reduce $\frac{x^2-y^2+4}{x+y}$ to an integral or a mixed quantity.

Ans. $x-y+\frac{4}{x+y}$.

9. Reduce $\frac{4ay-2y^2+a^3}{2a-y}$ to an integral or a mixed quantity.

Ans. $2y+\frac{a^3}{2a-y}$.

ADDITION OF FRACTIONS.

(94.) The Sum of two or more Fractions is found by means of a *common denominator.*

Thus the Sum of $\frac{a}{x}$ and $\frac{b}{x}$ is $\frac{a+b}{x}$.

For it is evident that a divided by x, added to b divided by x, makes the sum of a and b divided by x.

In other words, if each of the *parts* a and b be divided by x, the *whole* $a+b$ will be divided by x.

Hence we have

RULE XIII.

(95.) *For the Addition of Fractions.*

1. If the fractions have not a *common denominator*, reduce them to a common denominator.

2. Add the numerators together, and place the Sum, as a numerator, over their common denominator.

3. *Mixed quantities* may be added under the form of *improper fractions;* or the integral and the fractional parts may be added separately.

EXAMPLE.

To add together $2a+\frac{b}{x}$ and $3a-\frac{c}{y}$.

Reducing these mixed quantities to improper fractions, they become

$$\frac{2ax+b}{x} \text{ and } \frac{3ay-c}{y}.$$

Reducing these fractions to a common denominator, we have

$$\frac{2axy+by}{xy} \text{ and } \frac{3axy-cx}{xy}.$$

Placing the sum of these numerators over the common denominator, the result is,

$$\frac{5axy+by-cx}{xy}, \text{ equal to } 5a+\frac{by-cx}{xy}.$$

Otherwise, by adding the *integral* and the *fractional* parts separately.—There will be less liability to error, if we change the ***sign*** before the fraction in the second quantity to +, and change the sign of its numerator; thus

$$+\frac{-c}{y}\text{, (84).}$$

Then, reducing $\frac{b}{x}$ and $\frac{-c}{y}$ to a common denominator, they become

$$\frac{by}{xy} \text{ and } \frac{-cx}{xy}.$$

Adding these fractions together, and adding together the integral parts $2a$ and $3a$, we obtain

$$5a+\frac{by-cx}{xy}\text{, as before.}$$

Improper fractions in the results obtained by this Rule, should be reduced to integral or mixed quantities; and proper fractions, to their lowest terms.

EXERCISES.

1. Add together $\frac{3a}{4}$, $\frac{2a}{3}$, and $\frac{3a+2}{3}$.

Ans. $2a+\frac{5a+8}{12}$.

2. Add together $2x+\frac{a^2}{4}$, and $4x+\frac{2a^2}{5}$.

Ans. $6x+\frac{13a^2}{20}$.

3. Add together $y^2+\frac{2x}{3}$, and $3y^2-\frac{x}{4}$.

Ans. $4y^2+\frac{5x}{12}$.

4. Add together $\frac{a^2}{2x}$, $2a^2$, and $\frac{2b-3}{5}$.

Ans. $2a^2+\frac{5a^2+4bx-6x}{10x}$.

5. Add together $3x^2+\frac{a+1}{2}$ and $\frac{2a+3}{4}$.

Ans. $3x^2+a+\frac{5}{4}$.

6. Add together x^2y, $\frac{a^2}{3}$, and $x^2y+\frac{1}{2}$.

Ans. $2x^2y+\frac{2a^2+3}{6}$.

7 Add together $2a^2-\frac{x}{4}$ and $\frac{a^2-3x}{2}$.

Ans. $2a^2+\frac{2a^2-7x}{4}$.

8. Add together $\frac{a^2}{y}$, $\frac{3a}{y}$, and $\frac{a^2-4}{y^2}$.

Ans $\frac{a^2y+3ay+a^2-4}{y^2}$.

9 Add together $\frac{x-1}{3}$, 2, and $\frac{3x+4}{5}$.

Ans. $2+\frac{14x+7}{15}$

10. Add together $2y+\frac{2a^3}{3}$ and $y-\frac{3a^3}{4}$.

Ans. $3y-\frac{a^3}{12}$.

11. Add together $5a^2$, $\frac{3x}{c}$, and $\frac{a^2+1}{c^2}$.

Ans. $5a^2+\frac{3cx+a^2+1}{c^2}$.

12. Add together 10, $\frac{-x^2}{3}$, and $3-\frac{x^2}{5}$.

Ans. $13-\frac{8x^2}{15}$.

13. Add together $\frac{a^2+x}{3}$ and $\frac{a^2-x+1}{4}$.

Ans. $\frac{7a^2+x+3}{12}$.

14. Add together $2y^2-\frac{3}{a}$ and $\frac{3+y^2-x^2}{5}$.

Ans. $2y^2-\frac{15-3a-ay^2+ax^2}{5a}$

SUBTRACTION OF FRACTIONS.

(96.) The Difference of two Fractions is found by means of a *common denominator.*

Thus $\frac{a}{x}$ subtracted from $\frac{b}{x}$ leaves $\frac{b-a}{x}$.

For the last fraction, $\frac{b-a}{x}$, added to $\frac{a}{x}$ the one subtracted, produces $\frac{b}{x}$ the one from which the subtraction is made.

We have therefore

RULE XIV.

(97.) *For the subtraction of Fractions.*

1. If the fractions have not a *common denominator*, reduce them to a common denominator.
2. Subtract the numerator of the fraction to be subtracted from the other numerator, and place the Difference, as a numerator, over the common denominator.
3. A *mixed quantity* may be taken in subtraction under the form of an *improper fraction;* or the integral and the fractional part may be taken separately in subtracting.

EXAMPLE.

From $8x+\frac{a}{y}$ to subtract $3x-\frac{b}{w}$.

Reducing these mixed quantities to improper fractions, they become

$$\frac{8xy+a}{y} \text{ and } \frac{3xw-b}{w}.$$

Reducing these fractions to a common denominator,—subtracting the second of the resulting numerators from the first,—and placing the difference over the common denominator, we find

$$\frac{5xyw+aw+by}{yw}, \text{ equal to } 5x+\frac{aw+by}{yw}.$$

Otherwise, by taking the integral and the fractional parts separately.—To diminish the liability to error in adjusting the *signs*, we change the fraction in the second quantity to $+\dfrac{-b}{w}$, (84).

Then, reducing $\dfrac{a}{y}$ and $\dfrac{-b}{w}$ to a common denominator, they become

$$\frac{aw}{yw} \text{ and } \frac{-by}{yw}.$$

Subtracting the second of these fractions from the first, and $3x$ from $8x$, we find the difference of the given quantities to be

$$5x+\frac{aw+by}{yw}, \text{ as before.}$$

In all subsequent exercises, improper fractions in the results should be reduced to integral or mixed quantities; and proper fractions, to their lowest terms.

EXERCISES.

1. From $\dfrac{a^2+2}{3}$ subtract $\dfrac{b-3}{2a}$.
Ans. $\dfrac{2a^3+4a-3b+9}{6a}$.

2. From $\dfrac{3a+x}{y}$ subtract $\dfrac{a-x}{y^2}$.
Ans. $\dfrac{3ay+xy-a+x}{y^2}$.

3. From $\dfrac{a^2}{1+x}$ subtract $\dfrac{b}{1-x}$.
Ans. $\dfrac{a^2-a^2x-b-bx}{1-x^2}$.

4. From $3a+\dfrac{b}{x}$ subtract $2a-\dfrac{c}{y}$.
Ans. $a+\dfrac{by+cx}{xy}$.

5. From $2x^2-\dfrac{a}{x}$ subtract $x^2-\dfrac{b}{y}$.
Ans. $x^2-\dfrac{ay-bx}{xy}$.

6. From $\frac{1}{x-y}$ subtract $\frac{1}{x^2-y^2}$.

Ans. $\frac{x+y-1}{x^2-y^2}$.

7. From $3y^2-\frac{a}{y}$ subtract $2y^2+\frac{b}{x}$.

Ans. $y^2-\frac{ax+by}{xy}$.

8. From $2a+b+\frac{x}{2}$ subtract $3b-\frac{y}{2}$.

Ans. $2a-2b+\frac{x+y}{2}$.

9. From $4x^2-y-\frac{a^3}{3}$ subtract $y+\frac{c^3}{2}$.

Ans $4x^2-2y-\frac{2a^3+3c^3}{6}$.

10. From $2b^2+\frac{a+1}{2}$ subtract $b^2-\frac{a-2}{3}$.

Ans. $b^2+\frac{5a-1}{6}$.

11. From $5a^3+\frac{x}{y+1}$ subtract $3a^3+\frac{x}{y-1}$.

Ans. $2a^3-\frac{2x}{y^2-1}$.

12. From $3x^2-\frac{a+b}{2}$ subtract $-x^2+\frac{a^2}{3}$.

Ans. $4x^2-\frac{3a+3b+2a^2}{6}$.

13. From $2b^3-\frac{1+a}{3}$ subtract $b^2-\frac{1-a}{4}$.

Ans. $2b^3-b^2-\frac{1+7a}{12}$.

14. From $5+\frac{a^2+x^2}{y}$ subtract $1+\frac{a^2-x^2}{y^2}$.

Ans. $4+\frac{a^2y+x^2y-a^2+x^2}{y^2}$.

MULTIPLICATION OF FRACTIONS.

(98.) The Product of two or more Fractions is equal to the product of their numerators divided by the *product of their denominators*

Thus $\frac{a}{x^2}$ multiplied by $\frac{b}{y^3}$ produces $\frac{ab}{x^2y^3}$.

For the first of these fractions is equivalent to ax^{-2}, and the second to by^{-3}, (77); and the product of these *two equivalents* is

$$abx^{-2}y^{-3}, \ (40).$$

And by transferring the factors x^{-2} and y^{-3} to the *denominator*, we have

$$\frac{ab}{x^2y^3}, \ ab \text{ divided by } x^2y^3, \ (78).$$

From the preceding it follows, that

(99.) Multiplying by a Fraction finds *such a part of the multiplicand* as is expressed by the multiplier.

Thus $a \times \frac{1}{2}$ or $\frac{a}{1} \times \frac{1}{2}$ produces $\frac{a}{2}$, $\frac{1}{2}$ of a.

Compound Fractions.

(100.) A Fraction multiplied by *a fraction*, or divided by an *integer*, may be expressed by a *Compound fraction*, that is, a fraction of a fraction.

For example, $\frac{a}{x} \times \frac{1}{2}$ is $\frac{1}{2}$ of $\frac{a}{x}$, (99); also $\frac{a}{x} \div 2$ is $\frac{1}{2}$ of $\frac{a}{x}$.

Hence

(101.) Multiplying two or more Fractions together is equivalent to reducing a *compound* to a *simple* fraction.

Thus $\frac{a}{x} \times \frac{1}{2}$ is $\frac{1}{2}$ of $\frac{a}{x}$, equal to $\frac{a}{2x}$.

From proposition (98), we have the following Rule.

RULE XV

(102.) *For the Multiplication of Fractions.*

1. Multiply the numerators together for a *numerator*, and the denominators together for a *denominator*.

2. An *integral* quantity and a *fraction* are multiplied together, by multiplying the *numerator*, or dividing the denominator, by the integer.

3. A *mixed* quantity may be taken in multiplication under the form of an *improper fraction;* or the integral and the fractional part may be taken separately in multiplying.

It may also be remarked that

(103.) A Fraction is multiplied by *its own denominator*, by merely canceling the denominator.—And *equal factors may be canceled* in a numerator and its own or the other *denominator*, without altering the Product of the two fractions, (81).

EXAMPLE.

To multiply $\frac{a^2-x^2}{3y^2}$ by $\frac{2y}{a^2+2ax+x^2}$.

The first numerator is equal to $(a+x)(a-x)$, (58),
and the second denominator, to $(a+x)(a+x)$, (59),

By canceling the factor $a+x$ from these terms, and the factor y from the other numerator and denominator, the operation is reduced to

$$\frac{a-x}{3y} \times \frac{2}{a+x}; \text{ which produces } \frac{2a-2x}{3ay+3xy}.$$

EXERCISES.

1. Multiply together a^2-x^2 and $\frac{3}{a-x}$. *Ans.* $3a+3x$.

2. Multiply together a^2y and $x+\frac{3}{4}$. *Ans.* $a^2xy+\frac{3a^2y}{4}$.

3. Multiply together $\frac{a+x}{3}$ and $\frac{2}{a-x}$. *Ans.* $\frac{2a+2x}{3a-3x}$.

4. Multiply together $5y^2$ and $y^2-\frac{a}{3}$. *Ans.* $5y^4-\frac{5ay^2}{3}$.

5. Multiply together $\frac{a+b}{3x^2}$ and $\frac{a-b}{a+b}$. *Ans.* $\frac{a-b}{3x^2}$.

6 Multiply together $2a^2-3x$ and $\frac{3b}{y-1}$. *Ans.* $\frac{6a^2b-9bx}{y-1}$.

7. Multiply together $a+\frac{ax}{a-x}$ and $x-\frac{ax}{a+x}$. *Ans.* $\frac{a^2x^2}{a^2-x^2}$.

8. Multiply together $\frac{a^4-b^4}{a+b}$ and $\frac{a^2}{ab-b^2}$. *Ans.* $\frac{a^4+a^2b^2}{b}$.

9. Multiply together $5y^2-\frac{a}{x^2}$ and $5y^2+\frac{a}{x^2}$. *Ans.* $25y^4-\frac{a^2}{x^4}$.

10. Multiply together $\frac{3x^2-5x}{14}$ and $\frac{7a}{2x^3-3x}$. *Ans.* $\frac{3ax-5a}{4x^2-6}$.

DIVISION OF FRACTIONS.

(104.) The Quotient of two Fractions is equal to the dividend multiplied by the *reciprocal of the divisor*, (80).

Thus the Quotient of $\frac{a}{b}$ divided by $\frac{x}{y}$ is equal to $\frac{a}{b} \times \frac{y}{x}$.

For since the value of the dividend is not altered by multiplying each of its terms by both terms of the divisor, (81), the quotient is equal to

$$\frac{axy}{bxy} \div \frac{x}{y}.$$

And by dividing the numerator of this dividend by the numerator of the divisor, and the denominator by the denominator, we have the quotient $\frac{ay}{bx}$, which is equal to $\frac{a}{b} \times \frac{y}{x}$.

Observe that the Quotient *multiplied into the divisor*, produces the dividend, (46.)

Thus $\frac{ay}{bx} \times \frac{x}{y}$ produces $\frac{axy}{bxy}$, equal to $\frac{a}{b}$.

Complex or Mixed Fractions.

(105.) When the dividend or the divisor is a Fraction or a *mixed* quantity, the dividend over the divisor, with a line between them, forms a *complex* or *mixed* fraction.

Thus $\frac{a}{b} \div (x+y)$ equals $\frac{\frac{a}{b}}{x+y}$, numerator $\frac{a}{b}$, denominator $x+y$.

From the nature of Division, and the proposition before demonstrated, (104), we have

RULE XVI.

(106.) *For the Division of Fractions.*

1. Divide the numerator of the divisor into the numerator of the dividend, and the denominator into the denominator; or *multiply the dividend by the reciprocal of the divisor.*
2. A *fraction is divided by an integral* quantity, by dividing the numerator, or multiplying the denominator, by the integer.
3. An *integral quantity is divided by a fraction,* by dividing the integer by the numerator, and multiplying by the denominator; or by multiplying the integer by the *reciprocal of the fraction.*
4. A *mixed* quantity may be taken in division under the form of an *improper fraction:* or the integral and the fractional part may be divided separately

EXAMPLE.

To divide $\frac{10c^2}{a^2-2ax+x^2}$ by $\frac{5c^3}{a^2-x^2}$.

The dividend must be multiplied by $\frac{a^2-x^2}{5c^3}$, which is the reciprocal of the divisor.

If we cancel the factor $a-x$ from $a^2-2ax+x^2$, (60), and from a^2-x^2, (58); and also cancel the factor $5c^2$, the operation will be, (103),

$$\frac{2}{a-x} \times \frac{a+x}{c}; \text{ which produces } \frac{2a+2x}{ac-cx}.$$

By thus canceling common factors, we find the Quotient in its *lowest terms.*

EXERCISES.

1. Divide $2a^2-2y^2$ by $\frac{a+y}{5cx}$. *Ans.* $10acx-10cxy$

2 Divide $\frac{3a}{a^2-x^2}$ by $\frac{3}{a-x}$. *Ans.* $\frac{a}{a+x}$.

3. Divide $\frac{ax+b}{a}$ by $\frac{bx-a}{b}$. *Ans.* $\frac{abx+b^2}{abx-a^2}$.

4. Divide $\frac{a^2+2ab+b^2}{4ax^2}$ by $\frac{a+b}{2x}$. *Ans.* $\frac{a+b}{2ax}$

5. Divide $\frac{6a^2b}{x^2-2xy+y^2}$ by $\frac{3a^2}{x-y}$. *Ans.* $\frac{2b}{x-y}$

6 Divide $3a^2+\frac{c+x^2}{2}$ by $a^2-\frac{x^2}{3}$. *Ans.* $3+\frac{9x^2+3c}{6a^2-2x^2}$

7. Divide $\frac{2x^2}{a^3+x^3}$ by $\frac{x}{x+a}$. *Ans.* $\frac{2x}{x^2-ax+a^2}$

8. Divide $\frac{5a^2y-10ay^2+5a}{ax^2}$ by $\frac{5a}{x^2}$. *Ans.* $y-\frac{2y^2-1}{a}$.

9. Divide $\frac{a^3+b^3}{2+3x}$ by $ab+b^2$. *Ans.* $\frac{a^2-ab+b^2}{2b+3bx}$

10. Divide $2a^2+4ay+2y^2$ by $\frac{a+y}{2}$. *Ans.* $4a+4y$

11. Add $\frac{a}{3}$ to $\frac{x}{4}$, and divide the Sum by $\frac{3y}{4}$. *Ans.* $\frac{4a+3x}{9y}$.

12. Add $\frac{a-y}{5}$ to $\frac{a+y}{4}$, and divide the Sum by $\frac{x^2}{10}$. *Ans.* $\frac{9a+y}{2x^2}$

13. Add $-\frac{x^2}{2}$ to $\frac{2x^2-1}{3}$, and divide the Sum by $\frac{x^4-4}{2}$. *Ans.* $\frac{1}{3(x^2+2)}$.

14. Add $-\frac{ax}{2}$ to $\frac{2ax+1}{3}$, and divide the Sum by $\frac{(ax+2)^2}{3}$. *Ans.* $\frac{1}{2(ax+2)}$.

15. Subtract $\frac{y}{3}$ from $\frac{ay}{2}$, and divide the Remainder by $\frac{2x}{3}$. *Ans.* $\frac{3ay-2y}{4x}$.

16. Subtract $\frac{y^2}{4}$ from $\frac{a^2}{4}$, and divide the Remainder by $\frac{a+y}{2}$. *Ans.* $\frac{a-y}{2}$.

17. Subtract $\frac{x^2}{5}$ from $\frac{a^2}{5}$, and divide the Remainder by $\frac{a+x}{3}$. *Ans.* $\frac{3(a-x)}{5}$.

18. Multiply $\frac{2ab-3}{x}$ by $\frac{5a}{x}$, and divide the Product by $\frac{5}{x}$. *Ans.* $\frac{2a^2b-3a}{x}$.

19. Multiply $\frac{ax^2}{y^2}$ by $\frac{ay^2}{x^2}$, and divide the Product by $\frac{a+b}{3}$. *Ans.* $\frac{3a^2}{a+b}$.

20. Multiply $\frac{a-x}{6}$ by $\frac{a+x}{3}$, and divide the Product by $\frac{y}{9}$. *Ans.* $\frac{a^2-x^2}{2y}$.

21. Multiply $\frac{9+y^2}{2}$ by $\frac{9-y^2}{5}$, and divide the Product by $\frac{3-y}{10}$. *Ans.* $27+9y+3y^2+y^3$.

CHAPTER V.

SIMPLE EQUATIONS.

(107.) An EQUATION is an expression denoting the *equality of two quantities* by means of the sign $=$, *equal to*, placed between them.

The quantity on the left of the sign $=$ is called the *first member*, or *side*, and that on the right the second member, or side, of the Equation.

Thus $3x+ab=5x+8d-9$ is an equation,
in which $3x+ab$ is the first member, and $5x+8d-9$ is the second.

(108.) Equations are employed in the solution of particular mathematical questions, or in the investigation of general mathematical principles.

In the solution of questions, the *unknown* or required quantity is represented by a letter, usually x, or y, &c., and an Equation is then formed which expresses the relation between this and the *known* or given quantities.

To give a simple example;—Suppose we wish to find a number the *third* and *fourth* of which shall together make 35.

Let x represent the number to be found, and the Equation

will be $\frac{x}{3}+\frac{x}{4}=35$.

(109.) The *solution* of an Equation consists in finding the value of the *unknown quantity* in the equation.

The value found for the unknown quantity is *verified*, or the Equation *satisfied*, when this value, substituted for its symbol in the equation, makes the *first member the same as the second.*

The value of x in the preceding equation is 60, since this number, substituted for x, satisfies the equation; thus

$$\frac{60}{3}+\frac{60}{4}=35.$$

The mode of solution will vary with the

Different Degrees of Equations.

(110.) A *simple* Equation, or an equation of the *first degree*, is one which contains no power of the unknown quantity but its *first power.*

$$3x+ax-4=20 \text{ is a simple equation.}$$

A *quadratic* Equation, or an equation of the *second degree*, is one in which the highest power of the unknown quantity is its second power or *square.*

$$2x^2+3ax+5=30 \text{ is a quadratic equation.}$$

A *cubic* Equation, or an equation of the *third degree*, is one in which the highest power of the unknown quantity is its third power, or *cube;* and so on.

Equations are also distinguished as

Numerical, Literal, and Identical Equations.

(111.) A *numerical* Equation is one in which all the *known* quantities are expressed by *numbers.*

$$2x+5x=25-3 \text{ is a numerical equation.}$$

A *literal* Equation is one in which some or all of the *known* quantities are represented by *letters.*

$$2x+ax=25-3b \text{ is a literal equation,}$$

in which a and b are supposed to represent quantities whose values are *known.*

An *identical* Equation is one in which the two members *are the same*, or become the same by performing the operations which are indicated in them.

Thus $3x-3ab=3(x-ab)$ is an identical equation.

Transformation of Equations.

(112.) The *transformation* of an Equation consists in changing its *form,* without destroying the *equality* of the two members,—for the purpose of finding the value of the *unknown* quantity, or of discovering some general truth or principle.

These transformations depend, for the most part, on the following

Axioms.

(113.) An *Axiom* is a truth which is *self-evident,*—neither admitting nor requiring any *demonstration;* such as,

1. Things which are equal to the same thing, *are equal to each other.*
2. If equals be added to equals, the *sums will be equal.*
3. If equals be taken from equals, the *remainders will be equal.*
4. If equals be multiplied by equals, the *products will be equal.*
5. If equals be divided by equals, the *quotients will be equal.*
6. Any like *powers* or *roots* of equal quantities, *are equal*

SOLUTION OF SIMPLE EQUATIONS CONTAINING BUT ONE UNKNOWN QUANTITY.

(114.) The *value of the unknown* quantity is found by making its symbol stand alone on one side of the Equation, so as to be equal to known quantities on the other side.

In order to this, the following transformations may be necessary, or at least may be expedient.

1. Clearing the Equation of Fractions.
2. The Transposition and Addition of Terms.
3. Changing the Signs of all the Terms in the Equation.
4. Dividing the Equation by the Coefficient of the Unknown quantity.

We shall apply each of these transformations to the solution of the same Equation.

Clearing an Equation of Fractions.

(115.) An Equation is cleared of fractions by multiplying each numerator into all the denominators *except its own*—regarding each *integral* term as a numerator,—and omitting the given denominators

Let the Equation be

$$\frac{3x}{4}-7=x-\frac{28}{3}+\frac{x}{3}$$

Multiplying the numerator of each fraction by the denominators of the other two, and the integral terms 7 and x by all the denominators, we obtain

$$27x-252=36x-336+12x.$$

The equality of the two members is not destroyed in thus clearing the Equation of fractions, because each of the terms connected by the signs + and − in the two members, is thus multiplied by all the denominators. (103) (113...4).

An Equation may also be *cleared of fractions* by multiplying its two members by the *least common multiple* of the denominators;—observing that a fractional term will be multiplied by multiplying its numerator into the *quotient* of said multiple ÷ the denominator.

In the given Equation the least common multiple of the denominators is 12. Multiplying by 12, we find

$$9x-84=12x-112+4x.$$

The advantage of this method is, that the new equation is found in its *lowest terms*.

Transposition and Addition of Terms.

(116.) Any term may be transposed from one side of an Equation to the other by *changing its sign.*—All the similar terms may thus be placed on the same side, and then added together.

In the last Equation $9x-84=12x-112+4x$, by transposing -84 to the second member, and $12x$ and $4x$ to the first, we have

$$9x-12x-4x=-112+84;$$

And by adding together the similar terms,

$$-7x=-28.$$

The equality of the two members is not destroyed by transposing a term with *its sign changed* from one side to the other, because this is equivalent to adding the term with *its sign changed* to both sides.

Thus by adding 84 to both members of the equation

$$9x-84=12x-112+4x,$$

the term -84 is canceled in the first member (28). In like manner by adding $-12x$ to both members, $12x$ is canceled in the second member; so also with $4x$. (113...2).

From the preceding principles it follows, that

Two equal terms with like signs on opposite sides of the sign $=$, may be at once suppressed from the Equation.

Change of the Signs in an Equation.

(117.) All the signs in an Equation may be changed, $+$ to $-$ and $-$ to $+$, without affecting the equality of its two members.

This follows from the principle of Transposition, (116), since in transposing *all the terms*, the signs would all be changed, but the two members would still be equal.

In the Equation already found

$$-7x=-28,$$

we shall have, by Transposition,

$$28=7x, \text{ or } 7x=28.$$

The only Transformation which remains towards finding the *value* of x in the equation at first assumed, is that of dividing by the *coefficient* of x, the unknown quantity. (113...5).

Dividing both members of the preceding Equation by the coefficient of x, we find

$$x=\frac{-28}{-7}=4;\text{ or } x=\frac{28}{7}=4.$$

We have thus found the value of x to be 4. This value may be verified by substituting it for x in the original equation.

We may now give

RULE XVII.

(118.) *For the Solution of a Simple Equation containing but one unknown quantity.*

1. Clear the Equation of *fractions*, if it contains any.
2. Transpose all the terms containing the *unknown* quantity to one side, and all the known terms to the other side, of the equation.
3. Add together all the *similar terms* in each member.
4. Divide both members by the *coefficient* of the unknown quantity;—observing that when the unknown quantity is found in two or more *dissimilar* terms, its coefficient will be the sum of its coefficients in those terms.

NOTE.—When the sum of the terms containing the unknown quantity, after transposition, is *negative*, it will generally be expedient, though it is never necessary, to make it *positive* by changing all the signs in the equation.

EXAMPLE.

Given $\frac{x-3}{2}+\frac{x}{3}=20-\frac{x+19}{2}$, to find the value of x.

Clearing the equation of fractions, by multiplying it by the least common *multiple* of the denominator, which is 6, we have

$$3x-9+2x=120-3x-57.$$

By transposition,

$$3x+2x+3x=120-57+9.$$

Adding similar terms,

$$8x=72.$$

Dividing by the coefficient of x,

$$x=\frac{72}{8}=9.$$

Remark.—The student is apt to err in Clearing an Equation of its fractions, when, as in this Example, a fraction preceded by the negative sign has a *polynomial numerator.*

The sign $-$ before the fraction in the second member above, denotes that the fraction is to be *subtracted.* When this fraction is multiplied by the 6, the product $3x+57$ is subtracted by *changing its signs.* This gives the terms $-3x-57$ in the new equation.

EXERCISES.

Numerical Equations.

1. Given $4x-8=13-3x$ to find the value of x.

Ans. $x=3$.

2. Given $7x+17=10x-19$ to find the value of x.

Ans. $x=12$.

3. Given $8x+6=36-7x$ to find the value of x.

Ans. $x=2$.

4. Given $59-7x=4x+26$ to find the value of x.

Ans. $x=3$

5. Given $20-4x-12=92-10x$ to find the value of x.

Ans. $x=14$

6. Given $8-3x+12=30-5x+4$ to find the value of x.

Ans. $x=7$

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7. Given $\frac{x}{4}+24=\frac{3x}{2}$ to find the value of x.

Ans. $x=19\frac{1}{5}$.

8. Given $\frac{x}{2}+\frac{x}{3}=13-\frac{x}{4}$ to find the value of x.

Ans. $x=12$.

9. Given $\frac{x-5}{4}+6x=\frac{284-x}{5}$ to find the value of x.

Ans. $x=9$

10. Given $\frac{x+1}{2}+\frac{x+2}{3}=16-\frac{x+3}{4}$ to find the value of x.

Ans. $x=13$.

11. Given $3x-\frac{2x-5}{3}=x+\frac{x}{6}+13\frac{1}{3}$ to find the value of x.

Ans. $x=10$.

12. Given $\frac{12x+26}{5}-2x=15-\frac{x+3}{3}$ to find the value of x.

Ans. $x=12$

13. Given $\frac{19-x}{2}=x+\frac{11-x}{3}$ to find the value of x.

Ans. $x=5$.

14. Given $\frac{6x-10}{3}=\frac{18-4x}{3}+x$ to find the value of x

Ans. $x=4$.

15. Given $\frac{3x+4}{5}-\frac{7x-3}{2}=\frac{x-16}{4}$ to find the value of x.

Ans. $x=2$

16. Given $\frac{x+1}{3}-16=\frac{x+4}{5}-\frac{x+3}{4}$ to find the value of x.

Ans. $x=41$

17. Given $\frac{x-1}{7}=x-3-\frac{x+4}{3}$ to find the value of x.

Ans. $x=8$.

18. Given $\frac{4(x+2)}{3}=1+\frac{3x+1}{2}$ to find the value of x.

Ans. $x=7$.

19. Given $x-\frac{4x+2}{5}=3x-1\frac{1}{3}$ to find the value of x.

Ans. $x=\frac{1}{3}$.

20. Given $\frac{x}{2}-\frac{x}{3}+5=\frac{6(x+2)}{8}$ to find the value of x.

Ans. $x=6$.

21. Given $\frac{x}{3}+6x=\frac{4x-2}{5}$ to find the value of x.

Ans. $x=-\frac{6}{83}$.

22. Given $x-\frac{3x-5}{2}=12-\frac{2x-4}{3}$ to find the value of x.

Ans. $x=65$

23. Given $\frac{21-3x}{3}-\frac{2(2x+3)}{9}=6-\frac{5x+1}{4}$ to find the value of x.

Ans. $x=3$.

An Equation in which the unknown quantity is found in every term, with different *exponents* in different terms, may often be reduced to a *simple* Equation by dividing it by some *power of the unknown quantity*. (113...5).

Thus if $2x^3=10x^2$,—
by dividing by x^2 we have $2x=10$; hence $x=5$.

24. Given $\frac{x^2}{5}+3x=7x-\frac{3x^2}{5}$ to find the value of x.

Ans. $x=5$.

25. Given $\frac{3x}{4}-\frac{2x}{5}=\frac{x^2-10x}{2}$ to find the value of x.

Ans. $x=10\frac{7}{10}$.

26. Given $\frac{4}{3x}+\frac{5}{2x}=\frac{2}{x^2-10x}$ to find the value of x.

Ans. $x=10\frac{12}{23}$

Literal Equations.

27. Given $ax-c=\frac{x-b}{a+c}$ to find the value of x.

Clearing the equation of fractions, by multiplying it by the denominator $a+c$, we have

$$a^2x-ac+acx-c^2=x-b.$$

By transposition

$$a^2x+acx-x=ac+c^2-b.$$

Dividing by the coefficient of x,

$$x=\frac{ac+c^2-b}{a^2+ac-1}.$$

28. Given $x+\frac{ax}{c}=b-\frac{dx}{c}$ to find the value of x.

Ans. $x=\frac{bc}{a+c+d}$.

29. Given $bc-\frac{ab}{x}=-d-\frac{1}{x}$ to find the value of x.

Ans. $x=\frac{ab-1}{bc+d}$.

30. Given $3x-a=x-\frac{bx-d}{3}$ to find the value of x.

Ans. $x=\frac{3a+d}{6+b}$.

31. Given $4abx^2=\frac{3ax^2-2bx+ax}{3}$ to find the value of x.

Ans. $x=\frac{a-2b}{12ab-3a}$.

32. Given $\frac{x(a-b)}{2}=a+\frac{ab}{4}-\frac{x}{3}$ to find the value of x.

Ans. $x=\frac{3a(b+4)}{6(a-b)+4}$.

Remark.—In an *identical* Equation the unknown quantity has no *determinate value,* since any quantity whatever may be substituted for it, and the equation will be *satisfied.*

Thus in the equation

$$3x-5=3x-5,$$

the two members will be equal whatever be the value of x. (113...3)

PROBLEMS

In Simple Equations of one unknown Quantity.

(119.) A *Problem* is a question proposed for solution; and the solution of a problem by Algebra consists in *forming an Equation* which shall express the *conditions* of the problem, and then solving the equation.

The general method of *forming the Equation* of a problem, is, to represent a *required* quantity by x, or y, &c., and then to perform or indicate the same operations that would be necessary to *verify* the value of x or y, supposing that *value to have been found.*

EXAMPLES AND EXERCISES.

1. What number is that to the double of which if 13 be added, the sum will be 75?

Let x represent the required number;
then $2x$ will represent *twice the number;*
and, by the conditions of the problem, the equation will be

$$2x+13=75.$$

The value of x in this equation is the number required.

Ans. 31.

2. Find a number such that if it be multiplied by 5, and 24 be subtracted from the product, the remainder will be 36. *Ans.* 12.

3. What number is that to $\frac{1}{8}$ of which if 25 be added, the sum obtained will be equal to the number itself *minus* 39? *Ans.* 96.

4. Find a number such that if $\frac{1}{4}$ of it be subtracted from three times the number, the remainder will be 77. *Ans.* 28.

5. Find what number added to the sum of one half, one third, and one fourth of itself will equal 4 added to twice the number.

Ans. 48.

6. Divide the number 165 into two such parts that the less may be equal to $\frac{1}{10}$ of the greater.

Let x represent the *less part;*
then $165-x$ will represent the *greater;*
and the equation will be

$$x=\frac{165-x}{10}.$$

Ans. 15 and 150.

7. Divide the number 100 into two such parts that six times the less may be equal to twice the greater. *Ans.* 25 and 75.

8. It is required to divide 75 into two such parts that 3 times the greater may exceed 7 times the less by 15. *Ans.* 21 and 54.

9. What sum of money is that to which if $100 be added, $\frac{2}{3}$ of the amount will be $400? *Ans.* $500.

10. A prize of $100 is to be divided between two persons,—the share of the first being $\frac{7}{9}$ of that of the other. What are the shares? *Ans.* 43\frac{3}{4}$; 56\frac{1}{4}$.

11. A post is $\frac{1}{4}$ of its length in the mud, $\frac{1}{3}$ of it in the water, and 15 feet above the water. What is the length of the post? *Ans.* 36 feet.

12. Find a number such that if it be divided by 12, the divisor dividend and quotient together shall make 64. *Ans.* 48.

13. In a mixture of wine and cider, $\frac{1}{2}$ of the whole *plus* 25 gallons was wine, and $\frac{1}{3}$ part minus 5 gallons was cider. What was the whole number of gallons in the mixture? *Ans.* 120.

14. After a person had expended $10 more than $\frac{1}{3}$ of his money, he had $15 more than $\frac{1}{2}$ of it remaining. What sum had he at first? *Ans.* $150.

15. Divide the number 91 into two such parts that if the greater be divided by their difference, the quotient may be 7. *Ans.* 49 and 42.

16. A and B had equal sums of money; the first paid away $25, and the second $60, when it appeared that A had twice as much left as B. What sum had each? *Ans.* $95.

17. After paying away $\frac{1}{4}$ of my money, and then $\frac{1}{5}$ of what was left, I had $180. What sum had I at first? *Ans.* $300.

18. A line 37 feet in length is to be divided into 3 parts, so that the first may be 3 feet less than the second, and the second 5 more than the third; what are the parts? *Ans.* 12, 15, and 10 feet.

19. A can perform a piece of work in 12 days, and B can perform the same in 15 days. In what time could both together do the work?

Let x represent the number of days. Then since A could do $\frac{1}{12}$ of the work, and B $\frac{1}{15}$ of it, in 1 day,

$\frac{x}{12}$ will represent the part of the work A could do in x days,

$\frac{x}{15}$ will represent the part of the work B could do in x days

The equation is $\frac{x}{12} + \frac{x}{15} = 1$, the *entire work*. *Ans.* $6\frac{2}{3}$ days.

20. If A could mow a certain meadow in 6 days, B in 8 days, and C in 5 days, in what time could the three together do it?

Ans. $2\frac{2}{59}$ days.

21. Out of a cask of wine, which had leaked away a third part, 20 gallons were afterwards drawn, and the cask was then found to be but half full; how much did it hold? *Ans.* 120 gallons.

22. It is required to divide $300 between A, B, and C, so that A may have twice as much as B, and C as much as the other two together. *Ans.* A $100, B $50, C $150.

23. A gentleman spends $\frac{2}{3}$ of his yearly income in board and lodging, $\frac{2}{3}$ of the remainder in clothes, and then has $20 left. What is the amount of his income? *Ans.* $180.

24. A person at the time he was married, was 3 times as old as his wife, but 15 years afterwards he was only twice as old. What were their ages on their wedding day? *Ans.* 45 and 15 years.

25. Two persons, A and B, lay out equal sums of money in trade; the first gains $126, and the second loses $87, and A's money is now double of B's; what did each lay out? *Ans.* $300.

26. A courier, who travels 60 miles a day, had been dispatched 5 days, when a second is sent to overtake him, who goes 75 miles a day, in what time will he overtake him? *Ans.* 20 days.

27. An island is 60 miles in circumference, and two persons start together to travel the same way around it: A goes 15 miles a day; and B 20; in what time would the two come together again?

Ans. 12 days.

28. A man and his wife usually drank out a cask of beer in 12 days, but when the man was from home it lasted the woman 30 days; how many days would the man alone be in drinking it?

Let x be the number of days;

then $\frac{1}{x}$ is the part that he would drink in 1 day;

and since the woman would drink $\frac{1}{30}$ of it in 1 day, the equation will be

$$\frac{1}{x}+\frac{1}{30}=\frac{1}{12}, \text{ the part both would drink in 1 day.}$$

Ans. 20 days.

29. If A and B together can do a piece of work in 9 days, and A alone could do it in 15 days, in what time ought B alone to accomplish the work? *Ans.* $22\frac{1}{2}$ days.

30. The hour and the minute hand of a clock or watch are exactly together at 12 o'clock; when are they next together?

Ans. $5\frac{5}{11}$ minutes past *one*

31. It is required to divide $\$1000$ between A, B, and C, so that A shall have half as much as B, and C half as much as A and B together. *Ans.* $\$222\frac{2}{9}$; $\$444\frac{4}{9}$; $\$333\frac{1}{3}$.

32. A person being asked the hour, answered that the time past noon was $\frac{2}{3}$ of the time till midnight; what was the hour? *Ans.* 48 min. past 4.

33. It is required to divide the number 60 into two such parts that their product shall be equal to 3 times the square of the less; what are the parts? *Ans.* 15 and 45.

34. How much wine at 90 cents a gallon, and how much at $\$1.50$ a gallon, will be required to form a mixture of 20 gallons which shall be worth $\$1.25$ a gallon? *Ans.* $8\frac{1}{3}$ gal.; $11\frac{2}{3}$ gal.

35. A cistern is supplied with water by three pipes which would severally fill it in 4, 5, and 6 hours. In what time would three pipes running together fill the cistern? *Ans.* $1\frac{23}{37}$ hours.

36. If $\$1000$ be divided between A, B, and C, so that B shall have as much as A and half as much more, and C as much as B and half as much more, what will be the portion of each? *Ans.* $\$210\frac{10}{19}$; $\$315\frac{15}{19}$; $\$473\frac{13}{19}$.

37. A person has a lease for 99 years, and $\frac{2}{3}$ of the time which has expired on it is equal to $\frac{4}{5}$ of that which remains. Required the time which remains on the lease. *Ans.* 45 years.

38. A merchant bought cloth at the rate of $\$7$ for 5 yards, which he sold again at the rate of $\$11$ for 7 yards, and gained $\$100$. How many yards were thus bought and sold? *Ans.* $583\frac{1}{3}$ yards.

39. A and B together possess the sum of $\$9800$; and five-sixths of the sum owned by A is the same as four-fifths of that owned by B. What is the sum owned by each? *Ans.* $\$4800$; $\$5000$.

40. The assets of a bankrupt amounting to $\$5600$ are to be divided among his creditors A, B, and C, according to their respective claims. A's claim is $\frac{1}{2}$ of B's, and C's is $\frac{2}{3}$ of B's; what sum must each of the creditors receive? *Ans.* $\$1292\frac{4}{13}$; $\$2584\frac{8}{13}$; $\$1723\frac{1}{13}$.

Simple Equations Containing Two or more Unknown Quantities.

(120.) It is sometimes necessary to employ two or more unknown quantities in the solution of a Problem; and in this case there must be formed as many *independent* Equations as there are unknown quantities employed.

Two equations are said to be *independent* of each other when they express essentially *different conditions*, so that one of the equations is not a mere transformation of the other.

Equations which thus express different conditions of the same Problem, are sometimes called *simultaneous equations.*

Solution of Two Simple Equations Containing Two Unknown Quantities.

(121.) From two Equations containing two unknown quantities we may derive a *new equation* from which one of those quantities shall be *eliminated*, or made to *disappear*. The value of the remaining unknown quantity may be found from the new equation; and this value *put for its symbol* in one of the given equations, will determine the other unknown quantity.

Elimination by Addition or Subtraction.

(122.) The two terms which contain the *same letter* in two Equations, may be made *equal* by multiplying or dividing the equations by proper quantities. That letter will then be *eliminated* in the *sum*, or else in the *difference*, of the new equations.

EXAMPLE.

Given the equations $2x+3y=23$
and $5x-2y=10$,
to find the values of x and y.

Multiplying the first equation by 2, and the second by 3, we have
$4x+6y=46$,
and $15x-6y=30$. (113....4).

Adding together the corresponding sides of these equations, we find
$19x=76$. (113....2).
which gives $x=4$.

Putting 4 for x in the first of the two given equations, we obtain
$8+3y=23$,
which gives $y=5$.

Observe that if $6y$ had the *same sign* in the two equations which were *added together*, this term would have been eliminated by taking the *difference*, not the *sum*, of these equations.

Elimination by Substitution.

(123.) The value of one of the unknown quantities in an Equation, may be found in *terms of all the other quantities* in the equation. If this value be then substituted for its representative letter in another equation, that letter will be *eliminated.*

EXAMPLE.

Given, as before, $2x+3y=23$,
and $5x-2y=10$,
to find the values of x and y.

We will find the value of x in the first equation, as if the value of y were *known.*

By transposition, $2x=23-3y$;
dividing both members by the coefficient of x,

$$x=\frac{23-3y}{2}.$$

We now substitute this value of x for x in the second equation. In doing this, we must multiply this fraction by the coefficient 5 in the first term of that equation.

$$\text{Then } \frac{115-15y}{2}-2y=10.$$

Clearing this equation of fractions, transposing, &c., we shall find

$$y=5.$$

Putting 5 for y in the first equation,

$$2x+15=23;$$

which gives $x=4$.

The values of x and y are thus found to be the same as before.

This method of Elimination depends on the evident principle, that equivalent algebraic expressions may be taken, *the one for the other;* that is, *equal quantities may be substituted for each other.*

Elimination by Comparison.

(124.) If the value of the same letter be found in each of two Equations, in terms of the other quantities in the equations, that letter will be *eliminated* by putting one of these values *equal to the other.*

EXAMPLE.

Given, as before, $2x+3y=23$,
and $5x-2y=10$,
to find the values of x and y.

We will find the value of x in each equation, as if the value of y were known.

Transposing and dividing in the first equation,

$$x=\frac{23-3y}{2}.$$

Transposing and dividing in the second equation,

$$x=\frac{10+2y}{5}.$$

Putting the first of these values of x equal to the second,

$$\frac{23-3y}{2}=\frac{10+2y}{5}, (113....1).$$

which will give $y=5$.

By substituting 5 for y in any of the preceding equations, the value of x will be found to be 4, as in the two preceding solutions.

Before applying any of the preceding methods of Elimination, the Equations should generally be cleared of *fractions*, if they contain any; the necessary *transpositions* must be made; and similar terms must be added together.

Elimination by Addition or Subtraction will generally be found the simplest method, since it is free from *fractional expressions*, which are likely to occur in the application of the other two methods.

Let the student apply each of the three methods to the first ten of the following Exercises.

Equations may be marked, for reference, by the numbers (1), (2) (3), &c., or the capitals (A), (B), (C), &c.

Thus (A) $2x+y=10$, would be called equation (A).

EXERCISES.

1. Given $2x+3y=29$, and $3x-2y=11$, to find the values of x and y. *Ans.* $x=7$, and $y=5$.

2. Given $5x-3y=9$, and $2x+5y=16$, to find the values of x and y. *Ans.* $x=3$, and $y=2$.

3. Given $x+2y=17$, and $3x+y=16$, to find the values of x and y. *Ans.* $x=3$, and $y=7$.

4. Given $4x+y=34$, and $10y-x=12$, to find the values of x and y. *Ans.* $x=8$, and $y=2$.

5. Given $3x+4y=88$, and $6x+5y=128$, to find the values of x and y. *Ans.* $x=8$, and $y=16$.

6. Given $7x+3y=42$, and $8y-2x=50$, to find the values of x and y. *Ans.* $x=3$, and $y=7$.

7. Given $8y-3x=29$, and $6y-4x=20$, to find the values of x and y. *Ans.* $x=1$, and $y=4$.

8. Given $6x-5y=39$, and $7x-3y=54$, to find the values of x and y. *Ans.* $x=9$, and $y=3$.

9. Given $12x-9y=3$, and $12x+16y=228$, to find the values of x and y. *Ans.* $x=7$, and $y=9$.

10. Given $5x+7y=201$, and $8x-3y=137$, to find the values of x and y. *Ans.* $x=22$, and $y=13$.

11. Given $7y+\frac{x}{7}=99$, and $7x+\frac{y}{7}=51$, to find the values of x and y. *Ans.* $x=7$, and $y=14$.

12. Given $\frac{x}{2}+\frac{y}{3}=7$, and $\frac{x}{3}+\frac{y}{2}=8$, to find the values of x and y. *Ans.* $x=6$, and $y=12$.

13. Given $64-\frac{2x}{3}=\frac{4y}{5}$, and $77-\frac{5x}{6}=\frac{9y}{10}$, to find the values of x and y. *Ans.* $x=60$, and $y=30$.

14. Given $21-6y=\frac{x+8}{4}$, and $23-5x=\frac{y+6}{3}$, to find the values of x and y. *Ans.* $x=4$, and $y=3$.

15. Given $\frac{3x+4y}{5}=10-\frac{x}{4}$, and $\frac{6x-2y}{3}=14-\frac{y}{6}$, to find the values of x and y. *Ans.* $x=8$, and $y=4$.

16. Given $\frac{2x}{3}+\frac{3y}{5}=\frac{22}{5}$, and $\frac{3x}{5}+\frac{2y}{3}=4\frac{7}{15}$, to find the values of x and y. *Ans.* $x=3$, and $y=4$.

17. Given $\frac{x-2}{5}-\frac{10-x}{3}=\frac{y-10}{4}$, and $\frac{2y+4}{3}-\frac{2x+y}{8}=\frac{x+13}{4}$

to find the values of x and y. *Ans.* $x=7$, and $y=10$

18. Find the values of x and y in the equations

$$10x+55=10y+\frac{6x-35}{5}, \text{ and } \frac{80+3x}{15}, =18\tfrac{1}{3}-\frac{4x+3y-8}{7}.$$

Ans. $x=10$, and $y=15$.

19. Find the values of x and y in the equations

$$x-\frac{y-a}{b}=c, \text{ and } y-\frac{a-x}{b}=d.$$

Clearing the equations of fractions, we have

(A) $bx-y+a=bc$, and (B) $by-a+x=bd$.

By transposition in these two equations, we find

(C) $bx-y=bc-a$, and (D) $by+x=bd+a$.

Multiplying equation C by b, in order to eliminate y,

(E) $b^2x-by=b^2c-ab$.

Adding together equations D and E, (122.)

$$b^2x+x=b^2c+bd-ab+a.$$

Dividing both sides of this equation by the co-efficient of x,

$$x=\frac{b^2c+bd-ab+a}{b^2+1}.$$

The value of y will be found if we multiply equation B by b, subtract equation A, &c.

$$y=\frac{b^2d+ab-bc+a}{b^2+1}.$$

20. Given $\frac{x}{a}+\frac{y}{a}=1$, and $bx+cy=de$, to find the values of x and y.

Ans. $x=\frac{ac-de}{c-b}$, and $y=\frac{de-ab}{c-b}$.

21. Given $2cx-4c=-3dy$, and $7x=\frac{4b}{a}$, to find the values of x and y.

Ans. $x=\frac{4b}{7a}$, and $y=\frac{28ac-8bc}{21ad}$

22. Given $x+\frac{by}{a}=\frac{c}{a}$, and $m-\frac{ny}{x}=\frac{d}{x}$, to find the values of x and y

Ans. $x=\frac{nc+bd}{na+mb}$, and $y=\frac{mc-ad}{na+mb}$.

SOLUTION OF THREE OR MORE SIMPLE EQUATIONS CONTAINING AS MANY UNKNOWN QUANTITIES.

(125.) From three Equations containing three unknown quantities, we may derive two *new equations* containing but *two of the unknown* quantities—by eliminating one of the unknown quantities from the first and second equations, and the same unknown quantity from either of these and the third equation.

The two equations thus obtained may be solved as already exemplified, (121). Two of the unknown quantities will thus be determined; and by substituting their values in one of the given equations, the value of the other unknown quantity may be readily found.

In a similar manner four equations containing *four unknown* quantities, may be reduced to three equations containing but three unknown quantities; and these three may then be reduced to two Five equations may be reduced to *four*, these four to three, and these three to two, &c.

EXAMPLE.

Given the equations $x+y+z=29$,

$$x+2y+3z=62,$$

$6x+4y+3z=120$, to find x, y, and z.

Multiplying the first equation by 3, in order to eliminate z,

$$3x+3y+3z=87.$$

Subtracting this equation from the third equation,

$$3x+y=33;$$

Subtracting the second equation from the third,

$$5x+2y=58. \quad (122).$$

The last two equations contain but two unknown quantities, x and y, from which we may find $x=8$, and $y=9$.

Substituting these values of x and y in the first equation,

$$8+9+z=29,$$

which gives $z=12$.

If the value of x were found in the first equation, and substituted for x in the second and third equations, we should then have two new equations containing y and z, (123).

We might also have found the two *new equations* by finding the value of the same letter, as x, in each of the three given equations, then putting the first of these values equal to the second, and either the first or the second equal to the third, (124).

The best methods of elimination to be adopted in particular cases, can be learned only from experience. Regard should be had to simplicity and brevity in the operations employed.

EXERCISES.

23. Given $x+y+z=9$, $x+2y+3z=16$, and $x+3y+4z=21$, to find the values of x, y, and z. *Ans.* $x=4$, $y=3$, $z=2$.

24. Given $x+y+z=18$, $x+3y+2z=38$, and $x+\frac{1}{3}y+\frac{1}{2}z=10$, to find the values of x, y, and z. *Ans.* $x=4$, $y=6$, $z=8$.

25. Given $3x-9y=33$, $4y+2z=5x-20$, and $11x-7y=37+6z$, to find the values of x, y, and z. *Ans.* $x=2$, $y=-3$, $z=1$.

26. Given $2z=21-\frac{1}{3}(x+y)$, $3x=72$, and $38=\frac{1}{2}(3x+y-z)$ to find the values of x, y, and z *Ans.* $x=24$, $y=9$, $z=5$.

PROBLEMS

In Simple Equations of one, two, &c., Unknown Quantities

(126.) When two or more required quantities are so related that, when one of them is found, the others may be conveniently *derived from that one*, the Problem may be most readily solved by a single Equation. In some questions, however, it is necessary to represent each of the required quantities by an appropriate symbol, and then to form as many Equations as there are unknown quantities to be found.

EXAMPLES AND EXERCISES.

1. Find two numbers such that $\frac{1}{2}$ of the first with $\frac{1}{3}$ of the second shall be equal to 9, and $\frac{1}{4}$ of the first with $\frac{1}{5}$ of the second shall be equal to 5.

Let x represent the *first*, and y the *second* number, then by the conditions of the problem, we have

$$\frac{x}{2}+\frac{y}{3}=9, \text{ and } \frac{x}{4}+\frac{y}{5}=5.$$

The values of x and y in these equations are the numbers required. *Ans.* 8 and 15.

2. Divide the number 100 into two such parts that $\frac{1}{3}$ of the first and $\frac{1}{4}$ of the second part shall together make 30.

Ans. 60 and 40

3. Find two numbers such that their sum shall be 60, and the less number $\frac{1}{3}$ of the greater. *Ans.* 15 and 45.

4. At a certain election 946 men voted for two candidates, and the successful one had a majority of 558. How many votes were given for each candidate? *Ans.* 752 and 194.

5. Divide the number 48 into two such parts that the quotient of the greater part divided by 4, may be equal to 4 times the quotiet of the greater part divided by the less. *Ans.* 32 and 16.

6. A, B, and C make a joint contribution which in the whole amounts to $400; B contributes twice as much as A and $20 more, and C as much as the other two together. What sum did each contribute? *Ans.* A $60, B $140, C $200.

7. Find three numbers such that the sum of the 1st and 2d shall be 35, the sum of the 1st and 3d 40, and the sum of the 2d and 3d 45. *Ans.* 15, 20, and 25.

8. A sum of money was divided between A and B, so that B's share was $\frac{3}{5}$ of A's, and A's share exceeded $\frac{5}{9}$ of the whole sum by $50. What was the share of each? *Ans.* A's $450, B's $270.

9. The stock of three traders amounted to $760. The shares of the 1st and 2d together exceeded the share of the 3d by $240; and the share of the 1st was $360 less than the sum of the shares of the other two; what was the share of each?
Ans. $200; $300; $260.

10. A man being asked the age of himself and son, replied, "If I were $\frac{1}{4}$ as old as I am $+3$ times the age of my son, I should be 45; and if he were $\frac{1}{4}$ his present age $+3$ times mine, he would be 111." Required their ages. *Ans.* 36 and 12.

11. A and B together have $340, B and C together $384, and A and C together $356; what sum has each?
Ans. $156; $184; $200.

12. A number which is expressed by two digits is equal to 4 times the sum of its digits, and if 18 be added to the number, its digits will be interchanged with each other; what is the number?

Let x represent the *tens'*, and y the *units'* figure of the number; then $10x+y$ will represent the number.

By the conditions of the problem the equations will be

$$10x+y=4(x+y),$$
$$10x+y+18=10y+x.$$

Ans. 24.

13. It is required to divide the number 36 into three such parts, that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third, shall all be equal to each other; what are the parts? *Ans.* 8, 12, and 16.

14. A and B have both the same income; A saves $\frac{1}{5}$ of his annually, but B, by spending $50 per annum more than A, at the end of 4 years, finds himself $100 in debt; what is their income?
Ans. $125.

15. A gentleman purchased a chaise, horse, and harness for $180; the horse cost twice as much as the harness, and the chaise twice as much as the horse and harness together; what was the price of each?
Ans. $120; $40; $20.

16. A farmer purchased 100 acres of land for $2450; for a part of the land he paid $20 an acre, and for the other part $30 an acre. How many acres were there in each part?
Ans. 55, and 45 acres.

17. What fraction is that to the numerator of which if 1 be added, the value will be $\frac{1}{2}$; but if 1 be added to the denominator, the value of the fraction will be $\frac{1}{3}$? *Ans.* $\frac{3}{8}$.

18. A and B together possess an income of $570; if A's income were 3 times, and B's 5 times as much as each really is, they would together have $2350. What is the income of each?
Ans. $250; and $320.

19. How old are we? said a person to his father: 6 years ago, replied the latter, I was a third more than 3 times as old as you were; but in 3 years, if I multiply your age by $2\frac{1}{6}$, it will then be equal to mine. What were their ages? *Ans.* 15 and 36.

20. Find a number such that if we subtract it from 4980, divide the remainder by 8, and subtract 123 from the quotient, we shall find a remainder equal to the number itself. *Ans.* 444.

21. A laborer engaged for 40 days on these conditions; that for every day he worked he should receive 80 cents, but for every day he was idle he should forfeit 32 cents. At the end of the time he was entitled to $15.20; how many days did he work, and how many was he idle? *Ans.* 25, and 15 days.

22. A cistern containing 820 gallons is filled in 20 minutes by 3 pipes, the first of which conveys 10 gallons more, and the second 5 gallons less than the third, per minute. How much flows through each pipe in a minute? *Ans.* 22, 7, and 12 gallons.

23. A trader maintained himself for 3 years at an expense of £50 a year, and each year augmented that part of his stock which was not thus expended by $\frac{1}{3}$ thereof. At the end of the third year his original stock was doubled; what was that stock? *Ans.* £740.

24. A and B began to trade with equal sums of money. The first year A gained \$40, and B lost \$40; the second year A lost $\frac{1}{3}$ of what he had at the end of the first, and B gained \$40 less than twice what A lost; when it appeared that B had twice as much money as A. What sum did each begin with? *Ans.* \$320.

25. What fraction is that, whose numerator being doubled, and denominator increased by 7, the value becomes $\frac{2}{3}$; but the denominator being doubled, and the numerator increased by 2, the value becomes $\frac{3}{5}$? *Ans.* $\frac{4}{5}$.

26. A and B together can perform a piece of work in 8 days, A and C in 9 days, and B and C in 10 days. How many days would it take each person to perform the same work alone?

Let x, y, and z represent the number of days required for A, B, and C respectively;

Then $\frac{1}{x}$ is the part of the work that A could do in 1 day, &c., and, by the conditions of the problem, the equations will be

$$\frac{1}{x}+\frac{1}{y}=\tfrac{1}{8},\quad \frac{1}{x}+\frac{1}{z}=\tfrac{1}{9},\quad \frac{1}{y}+\frac{1}{z}=\tfrac{1}{10}.$$

By subtracting the second equation from the first, we shall eliminate x, and then by adding the third equation we shall eliminate z

Ans. A $14\frac{34}{49}$ days, B $17\frac{23}{41}$, C $23\frac{7}{31}$.

27. From two places, which are 154 miles apart, two persons set out at the same time to meet each other, one traveling at the rate of 3 miles in 2 hours, and the other at the rate of 5 miles in 4 hours; in how many hours will they meet? *Ans.* 56 hours.

28. In a naval engagement, the number of ships captured was 7 more, and the number burned was 2 less, than the number sunk. Fifteen escaped, and the fleet consisted of 8 times the number sunk; of how many ships did the fleet consist? *Ans.* 32.

29. A and B together could have completed a piece of work in 15 days, but after laboring together 6 days, A was left to finish it alone, which he did in 30 days. In how many days could each have performed the work alone? *Ans.* 50, and $21\frac{3}{7}$ days.

30. On comparing two sums of money it is found, that $\frac{2}{5}$ of the first is \$96 less than $\frac{3}{4}$ of the second, and that $\frac{5}{8}$ of the second is as much as $\frac{4}{9}$ of the first. What are the sums?

Ans. \$720, and \$512.

31. A privateer, running at the rate of 10 miles an hour, discovers a ship 18 miles off, making way at the rate of 8 miles an hour. In how many hours will the ship be overtaken? *Ans.* 9 hours.

32. In a composition of copper, tin, and lead, $\frac{1}{2}$ of the whole *minus* 16 pounds was copper, $\frac{1}{3}$ of the whole *minus* 12 pounds was tin, and $\frac{1}{4}$ of the whole *plus* 4 pounds was lead; what quantity of each was there in the composition? *Ans.* Copper 128, tin 84, lead 76 pounds.

33. The sum of \$660 was raised for a certain purpose by four persons, the first giving $\frac{1}{2}$ as much as the second, the third as much as the first and second, and the fourth as much as the second and third. What were the several sums contributed?

Ans. \$60, \$120, \$180, \$300.

34. Two pedestrians start from the same point, and go in the same direction; the first steps twice as far as the second, but the second makes 3 steps while the first is making 2. How far has each one gone when the first is 300 feet in advance of the second?

Ans. 1200, and 900 feet.

35. A merchant has cloth at \$3 a yard, and another kind at \$5 a yard. How many yards of each kind must he sell, to make 100 yards which shall bring him \$450? *Ans.* 25, and 75 yards.

36. In the composition of a quantity of gunpowder, the nitre was 10 pounds more than $\frac{2}{3}$ of the whole, the sulphur $4\frac{1}{2}$ pounds less than $\frac{1}{6}$ of the whole, and the charcoal 2 pounds less than $\frac{1}{7}$ of the nitre. What was the amount of gunpowder? *Ans.* 69 pounds.

37. Four places are situated in the order of the letters A, B, C, D. The distance from A to D is 34 miles; the distance from A to B is $\frac{2}{3}$ of the distance from C to D; and $\frac{1}{4}$ of the distance from A to B, *plus* $\frac{1}{2}$ of the distance from C to D, is 3 times the distance from B to C. What are the distances between A and B, B and C, C and D?

Ans. 12, 4, and 18 miles.

38. A vintner sold at one time 20 dozen of port wine, and 30 of sherry, for \$120; and at another time 30 dozen of port, and 25 of sherry, at the same prices as before, for \$140. What was the price of a dozen of each sort of wine? *Ans.* \$3, and \$2.

39. A person pays, at one time, to two creditors, \$53, giving to one of them $\frac{4}{11}$ of the sum due to him, and to the other \$3 more than $\frac{1}{6}$ of his debt to him. At another time he pays them \$42, giving to the first $\frac{3}{7}$ of what remains due to him, and to the other $\frac{1}{3}$ of what remains due to him. What were the debts? *Ans.* \$121, and \$36.

40. A farmer has 86 bushels of wheat at 4*s.* 6*d.* per bushel, with which he wishes to mix rye at 3*s.* 6*d.* per bushel, and barley at 3*s.* per bushel, so as to make 136 bushels, that shall be worth 4*s.* a bushel. What quantity of rye and of barley must he take?

Ans. 14, and 36 bushels.

41. A composition of copper and tin, containing 100 cubic inches, weighs 505 ounces. How many ounces of each metal does it contain, supposing the weight of a cubic inch of copper to be $5\frac{1}{4}$ ounces, and of a cubic inch of tin $4\frac{1}{4}$ ounces? *Ans.* 420, and 85 ounces.

42. A General having lost a battle, found that he had only one-half of his army *plus* 3600 men left, fit for action; $\frac{1}{8}$ of his men *plus* 600 being wounded, and the rest, who were $\frac{1}{5}$ of the whole army, either slain, taken prisoners, or missing. Of how many men did his army consist? *Ans.* 24000.

43. Two pipes, one of them running 5 hours, and the other 4, filled a cistern containing 330 gallons; and the same two pipes, the first running 2 hours, and the second 3, filled another cistern containing 195 gallons. How many gallons did each pipe discharge per hour? *Ans.* 30 and 45 gallons.

44. After A and B had been employed on a piece of work for 14 days, they called in C, by whose aid it was completed in 28 days. Had C worked with them from the beginning, the work would have been accomplished in 21 days. In how many days would C alone have accomplished the work? *Ans.* 42 days.

45. Some smugglers discovered a cave which would exactly hold their cargo, viz., 13 bales of cotton and 33 casks of wine. A revenue cutter coming in sight while they were unloading, they sailed away with 9 casks and 5 bales, leaving the cave two-thirds full. How many bales or casks would it contain? *Ans.* 24 bales or 72 casks.

46. A gentleman left a sum of money to be divided among four servants, so that the share of the first was $\frac{1}{2}$ the sum of the shares of the other three; the share of the second $\frac{1}{3}$ of the sum of the other three; and the share of the third $\frac{1}{4}$ of the sum of the other three; and it was also found that the share of the first exceeded that of the last by $14. What was the whole sum? and the share of each?

Ans. Whole sum $120; shares $40; $30; $24; $26.

CHAPTER VI.

RATIO—PROPORTION—VARIATION.

RATIO.

(127.) The RATIO of one quantity called the *antecedent* to another of the same kind called the *consequent*, is the *quotient* of the former divided by the latter.

Thus the *ratio* of 12 to 4 is 3, since 12 is 3 *times* 4;

and the ratio of 5 to 13 is $\frac{5}{13}$, since 5 is five *thirteenths* of 13.

The antecedent and consequent together are called the *terms* of the ratio.

Sign of Ratio.

(128.) A *colon* (:) between two quantities denotes that the two quantities are taken as the *antecedent* and *consequent* of a ratio.

Thus $3 : 5$, the ratio of 3 to 5; $a : b$, the ratio of a to b.

(129.) The *value* of a ratio may always be represented by making the antecedent the *numerator*, and the consequent the *denominator* of a Fraction.

Thus $3 : 5$ is equal to $\frac{3}{5}$; and $a : b$ is equal to $\frac{a}{b}$, (75).

Direct and Inverse Ratio.

(130.) The *direct ratio* of the first of two quantities to the second, is the quotient of the *first divided by the second;* thus the direct ratio of 3 to 5 is $\frac{3}{5}$.

The *inverse ratio* of the first quantity to the second, is the direct ratio of the *second to the first;*—in other words, it is the direct ratio of the *reciprocals of the two quantities.*

Thus the *inverse ratio* of 3 to 5 is $\frac{5}{3}$;—

or it is the ratio of $\frac{1}{3}$ to $\frac{1}{5}$, equal to $\frac{1}{3} \div \frac{1}{5}$, equal to $\frac{5}{3}$, (127).

Hence *inverse* is often called *reciprocal* ratio. The term *ratio* used alone always means *direct* ratio.

Compound Ratio.

(131.) A *compound ratio* is the ratio of the *product* of two or more antecedents to the product of their consequents; and is equal to the product of all the *simple ratios.*

The compound ratio of a and b to x and y is $\dfrac{ab}{xy}$;

$$= \text{the product of the simple ratios } \frac{a}{x} \text{ and } \frac{b}{y}, \text{ or } \frac{a}{y} \text{ and } \frac{b}{x}.$$

(132.) The ratio of the *first to the last* of any number of quantities, is equal to the product of the ratios of the first to the second, the second to the third, and so on to the last; that is, it is *compounded of all the intervening ratios.*

For example, take the quantities a, b, x, y. The ratios of the first to the second, the second to the third, &c., are

$$\frac{a}{b}, \frac{b}{x}, \frac{x}{y}; \text{ and their product is } \frac{abx}{bxy}, = \frac{a}{y}, \text{ which is } a : y.$$

Duplicate and Triplicate Ratios.

(133.) The *duplicate ratio* of two quantities is the ratio of their *squares,* and the *triplicate* ratio is the ratio of their *cubes.*

Thus the duplicate ratio of a to b is the ratio of a^2 to b^2;
and the triplicate ratio of a to b is the ratio of a^3 to b^3.

The *subduplicate* ratio of quantities is the ratio of their *square roots,* and the *subtriplicate* ratio is the ratio of their *cube roots*

Equimultiples and Equisubmultiples.

(134.) *Equimultiples* of two quantities are the products which arise from multiplying the quantities by the same *integer,* and *equisubmultiples* are the *quotients* which arise from dividing the quantities by the same *integer.*

Thus $3a$ and $3b$ are equimultiples of a and b, while, conversely, a and b are equisubmultiples of $3a$ and $3b$.

(135.) *Equimultiples,* or *equisubmultiples,* of two quantities have the same ratio as the quantities themselves, (81).

PROPORTION.

(136.) PROPORTION consists in an equality of the *ratios* of two or more antecedents to their respective consequents—but is usually confined to *four terms.*

(137.) *Four quantities* are in Proportion when the ratio of the *first* to the *second* is equal to the ratio of the third to the fourth; that is, when the first *divided by the second* is equal to the third divided by the fourth.

Thus the numbers 6, 3, 8, 4 are in proportion,
since the ratio $\frac{6}{3}$ equals the ratio $\frac{8}{4}$.

And the quantities a, b, x, y are in proportion,
when the ratio $\dfrac{a}{b}$ equals the ratio $\dfrac{x}{y}$, (129).

The first and third terms are the *antecedents* of the ratios; the second and fourth are the *consequents.* The first and fourth are the two *extremes;* the second and third are the two *means.*

The *fourth term* is called a fourth *proportional* to the other three taken in order; thus 4 is a fourth proportional to 6, 3, and 8.

(138.) *Three quantities* are in Proportion when the ratio of the first to the second is equal to the ratio of the *second to the third,*—the second term being called a *mean proportional* between the other two.

Thus the numbers 8, 4, 2 are in proportion,
since the ratio $\frac{8}{4}$ equals the ratio $\frac{4}{2}$;
and 4 is a *mean proportional* between 8 and 2.

Direct and Inverse Proportion.

(139.) A *direct* Proportion consists in an equality between two *direct* ratios, and an *inverse* or reciprocal Proportion in an equality between a direct and an inverse ratio.

Thus the numbers 6, 3, 8, 4 are in *direct* proportion; (137).

The same numbers in the order 6, 3, 4, 8 are in *inverse* proportion, since the direct ratio $\frac{6}{3}$ is equal to the *inverse* ratio $\frac{8}{4}$, (130).

The term *proportion* used alone always means *direct* proportion.

Sign of Proportion.

(140.) A Proportion is denoted by a *double colon* (: :), or the sign $=$ between the equal ratios of the proportion.

Thus $6 : 3 :: 8 : 4$, or $6 : 3 = 8 : 4$, or $\frac{6}{3} = \frac{8}{4}$

denotes that these numbers are in proportion, and is read

6 is to 3 as 8 is to 4.

To denote an *inverse* Proportion we employ the sign $\neq$ between the two ratios of such proportion.

Thus $6 : 3 \neq 4 : 8$, denotes that 6 is to 3 *inversely* as 4 is to 8.

Inverse Converted Into Direct Proportion.

(141.) An *inverse* is converted into a *direct* Proportion by interchanging either antecedent and its consequent; or by substituting the *reciprocals* of either antecedent and its consequent.

Thus from the *inverse* proportion $6 : 3 \neq 4 : 8$,
we get the *direct* proportion $3 : 6 = 4 : 8$, by interchanging 6 and 3,
or $\frac{1}{6} : \frac{1}{3} = 4 : 8$, by substituting $\frac{1}{6}$ and $\frac{1}{3}$.

The reason of this is evident from the nature of inverse ratio, (130).

VARIATION.

(142.) VARIATION is such a dependence of one term or quantity on another, that any *new value* of one of them will produce a new value of the other, in a constant ratio of increase or diminution.

1. One quantity *varies directly* as another when their dependence is such that if one of them be *multiplied*, the other must be multiplied, by the same quantity.

For example, the Interest on money, for a given *time* and *rate* per cent., *varies directly* as the Principal, since the Interest will be doubled, or tripled, &c., if the Principal be doubled, or tripled, &c.

2. One quantity *varies inversely* as another when their dependence is such that if one of them be multiplied, the other must be *divided*, by the same quantity.

For example, the Time in which a given amount of *interest* will accrue on a given *principal*, varies *inversely* as the Rate per cent., since the Time will be doubled, &c. if the Rate be *halved*, &c.

(143.) When one quantity *varies inversely* as another, the *product* of the two is always the same *constant quantity.*

For as one of the two quantities is *multiplied*, the other is *divided* by the same number; the product of the two will therefore be multiplied and divided by the same number; hence its value will remain *unchanged.*

Variation—an Abbreviated Proportion.

(144.) The two terms of a variation are the two *antecedents* in a Proportion in which the two consequents are not expressed, but may be understood, to complete the proportion.

Thus when we say that the Interest varies as the Principal, for a given *time* and *rate* per cent., it is understood, that

The Interest on any principal *is to* the Interest on any other Principal, for the same time and rate, *as* the *first* Principal *is to* the second.

Instead of saying "the Interest varies as the Principal," we may say, the Interest is *proportional* to the Principal; which is a brief method of expressing a Proportion by means of its *antecedents*,—the consequents being understood.

Sign of Variation.

The character $\sim$ placed between two terms, denotes that one of them *varies as the other.* Thus $x \sim y$, x varies as y, or x is proportional to y.

$x \sim \frac{1}{y}$ denotes that x varies as the *reciprocal* of y; or x varies reciprocally or *inversely* as y.

$x \sim \frac{y}{z}$ denotes that x varies *directly* as y, and *inversely* as z; that is, x varies as the *quotient* of y divided by z.

THEOREMS IN PROPORTION.

(145.) A *Theorem* is a proposition to be demonstrated or proved.—

A *Corollary* is an inference drawn from a preceding proposition or demonstration.

THEOREM I.

(146.) Two Fractions having a *common denominator*, are to each other as *their numerators;* and two fractions having a common numerator are to each other *inversely as their denominators.*

First. Let d be the common denominator;

then the ratio of $\frac{a}{d}$ to $\frac{c}{d}$ is

$$\frac{a}{d} \div \frac{c}{d} = \frac{ad}{cd} = \frac{a}{c}, \text{ (127)},$$

and $\frac{a}{c}$ is the ratio of the numerator a to the numerator c.

Secondly. Let n be the common numerator;

then the ratio of $\frac{n}{a}$ to $\frac{n}{c}$ is

$$\frac{n}{a} \div \frac{n}{c} = \frac{cn}{an} = \frac{c}{a};$$

and $\frac{c}{a}$ is the *inverse ratio* of the denominator a to the denominator c, (130).

Therefore, two fractions having a common denominator, &c.

(147.) *Corollary.* The value of a Fraction varies *directly as its numerator*, and *inversely as its denominator.*

THEOREM II.

(148.) In any Proportion, if one antecedent be greater than its consequent, the other antecedent will be *greater than its consequent;* if equal, equal; and if less, less.

Let $a : b :: x : y$;

then $\frac{a}{b} = \frac{x}{y}$, (137).

Now if a be greater than b the first ratio will be greater than a unit, and consequently the second ratio will be greater than a unit, and therefore x will be greater than y. In like manner if a be equal to b, x will be equal to y, &c.

Hence, in any Proportion, if one antecedent, &c.

THEOREM III.

(149.) When four quantities are in Proportion, the product of the two extremes *is equal to the product of the two means.*

Let $a : b :: x : y$;
then is $ay = bx$.
For since the quantities are in proportion,

$$\frac{a}{b} = \frac{x}{y}, \quad (137).$$

Clearing the Equation of fractions.

$$ay = bx. \quad (115).$$

Therefore, when four quantities are in Proportion, &c.

(150.) *Cor.* 1. A *fourth proportional* to three given quantities, is found by dividing the product of the *second and third* by the first.

Thus from the equation $ay = bx$, we find $y = \dfrac{bx}{a}$.

(151.) *Cor.* 2. When three quantities are in Proportion, the product of the two extremes *is equal to the square of the mean.*

For let $a : b :: b : x$; then $ax = bb = b^2$.

(152.) *Cor.* 3. A *mean proportional* between two given quantities, is equal to the *square root of their product.*

Thus from the equation $ax = b^2$, we find $b = (ax)^{\frac{1}{2}}$.

THEOREM IV.

(153.) When the product of two quantities is equal to the product of two other quantities, either pair of factors *may be made the extremes,* and the other the *means,* of a Proportion.

Let $ab = xy$;
then will $a : x :: y : b$.
Dividing both sides of the given equation by b,

$$a = \frac{xy}{b}.$$

Dividing both sides of this last equation by x,

$$\frac{a}{x} = \frac{y}{b}.$$

Hence $a : x :: y : b$, (137).

In like manner a and b may be taken for the *means,* and x and y for the *extremes.* Therefore, when the product of two quantities, &c

THEOREM V.

(154.) If three quantities are in Proportion, the first will be to the *third* as the *square of the first* to that of the second, or the square of the second to that of the third.

Let $a : b :: b : x$;

then will $a : x :: a^2 : b^2$; or $a : x :: b^2 : x^2$.

From the given proportion, we find

$ax = b^2$, (151).

Multiplying both sides of this equation by a,

$a^2x = ab^2$.

Converting this equation into a Proportion,

$a : x :: a^2 : b^2$, (153).

And by multiplying both sides of the first equation by x, we shall, in like manner, find $a : x :: b^2 : x^2$.

Therefore, if three quantities are in proportion, &c.

THEOREM VI.

(155.) Four quantities in Proportion are also in proportion by *inversion*,—that is, when each antecedent and its consequent are *interchanged with each other.*

Let $a : b :: x : y$;

then is $b : a :: y : x$.

From the given proportion we find

$ay = bx$, (149).

Making b and x the extremes, and a and y the means,

$b : a :: y : x$, (153).

Hence, four quantities in proportion are also in proportion, &c.

THEOREM VII.

(156.) Four quantities in Proportion are also in proportion by *alternation*,—that is, when the two *means*, or the two extremes, are *interchanged with each other.*

Let $a : b :: x : y$;

then is $a : x :: b : y$; or $y : b :: x : a$.

From the given proportion we find

$ay = bx$, (149).

This equation may be converted into the proportion

$a : x :: b : y$, or $y : b :: x : a$, (153).

Therefore, four quantities in proportion, &c.

THEOREM VIII.

(157.) In any Proportion, if the *two antecedents*, or the two consequents, or an antecedent and its consequent, be *multiplied by the same quantity*, the products and the remaining terms will be in proportion.

Let $a : b :: x : y$;
and let n be any numerical quantity;
then will $an : b :: nx : y$, &c.

From the given proportion we have

$$ay=bx.$$

Multiplying both sides of this equation by n,

$$any=bnx.$$

Converting this equation into a Proportion,

$$an : b :: nx : y;\ \text{or}\ a : bn :: x : ny;\ \text{or}\ a : b :: nx : ny,\quad (153).$$

Therefore, in any proportion, if the two antecedents, &c.

(158.) *Cor.* If the two *antecedents*, or the two consequents, or an antecedent and its consequent, be *divided by the same quantity*, the quotients and the remaining terms will be in proportion.

For dividing by a quantity is equivalent to *multiplying by its reciprocal.*

THEOREM IX.

(159.) Four quantities in Proportion are also in proportion by *composition*,—that is, the *sum of the first* and second terms is to the *first* or second, as the *sum of the third* and fourth is to the *third* or fourth

Let $a : b :: x : y$
then is $a+b : a :: x+y : x$.

From the given proportion we have

$$ay=bx.$$

Adding both sides of this equation to ax,

$$ax+ay=ax+bx.$$

Resolving each member of this equation into its factors,

$$a(x+y)=x(a+b).$$

Converting this last equation into a Proportion,

$$a+b : a :: x+y : x,\quad (153).$$

By adding both sides of the first equation to by, it may be proved, in like manner, that $a+b : b :: x+y : y$.

Therefore, four quantities in proportion are also in proportion, &c

THEOREM X.

(160.) Four quantities in Proportion are also in proportion **by *division*,**—that is, the *difference of the first* and second terms is to the ***first*** or second, as the *difference of the third* and fourth is to the ***third* or** fourth.

Let $a : b :: x : y$;

then is $a-b : a :: x-y : x$.

From the given proportion we find

$$ay=bx.$$

Subtracting both sides of this equation from ax,

$$ax-ay=ax-bx.$$

Resolving each member of this equation into its factors,

$$a(x-y)=x(a-b).$$

Converting this last equation into a Proportion,

$$a-b : a :: x-y : x.$$

By subtracting both sides of the first equation from by, it may be proved, in like manner, that $a-b : b :: x-y : y$.

Hence, four quantities in proportion are also in proportion, &c.

THEOREM XI.

(161.) When *any number of quantities* are in Proportion, the sum of any *two or more of the antecedents* is to the sum of their consequents, as any one antecedent is to its consequent.

Let $a : b :: c : d :: x : y$, &c.;

then is $a+c : b+d :: x : y$.

From the given proportion we shall find

$ay=bx$, and $cy=dx$, (149).

Adding together the corresponding members of these two equations,

$$ay+cy=bx+dx.$$

Resolving each member of this equation into its factors,

$$(a+c)y=(b+d)x,$$

Converting this equation into a Proportion,

$$a+c : b+d :: x : y.$$

By adding xy to both sides of the third equation, we shall, in like manner, find that $a+c+x : b+d+y :: x : y$.

Therefore, when any number of quantities are in proportion, &c.

THEOREM XII.

(162.) If two Proportions have an *antecedent* and *its consequent*, or the two antecedents, or the two consequents, the *same in both*, the remaining terms will be in proportion.

$$\begin{aligned} \text{Let } a : b &:: x : y, \\ \text{and } a : b &:: w : z; \\ \text{then will } x : y &:: w : z. \end{aligned}$$

For the ratio of x to y is equal to the ratio of w to z, since each of these ratios is equal to the ratio of a to b; hence

$$x : y :: w : z.$$

The two given proportions have *an antecedent* and its consequent *the same in both.* If the two antecedents were the same in both, the demonstration would be the same, after *interchanging the means;* and if the two consequents were the same, after interchanging the extremes, (156).

Hence, if two proportions have an antecedent and its consequent, &c.

Two or more Proportions having an antecedent and its consequent the *same in each*, form one *continued proportion.*

$$\begin{aligned} \text{Thus, } a : b &:: u : v, \\ a : b &:: w : x, \\ \text{and } a : b &:: y : z, \end{aligned}$$

form the continued proportion, $a : b :: u : v :: w : x : y : z$.

THEOREM XIII.

(163.) The sum of the *first* and *second* terms in any Proportion, is to their *difference*, as the sum of the third and fourth is to their difference.

$$\begin{aligned} \text{Let } a : b &:: x : y, \\ \text{then is } a+b : a-b &:: x+y : x-y. \end{aligned}$$

By Composition and Division, in the given proportion.

$$\begin{aligned} a+b : a &:: x+y : x; \quad (159); \\ a-b : a &:: x-y : x. \quad (160). \end{aligned}$$

These two proportions have the consequents a and x the same in both; hence $a+b : a-b :: x+y : x-y$, (162).

Therefore, the sum of the first and second terms in any proportion, &c.

THEOREM XIV.

(164.) The *products of the corresponding terms* of two or more Proportions, are in proportion.

Let $a : b :: x : y$,
and $c : d :: w : z$;
then is $ac : bd :: xw : yz$.

From the two given proportions, we have
$$ay = bx, \text{ and } cz = dw.$$
Multiplying together the corresponding members of these equations,
$$ac . yz = bd . xw.$$
Converting this equation into a Proportion,
$$ac : bd :: xw : yz, \quad (153).$$

In like manner the demonstration may be extended to three or more proportions. Hence, the products of the corresponding terms, &c.

(165.) *Cor.* Like *powers* or *roots* of proportional quantities, are in proportion.

For if $a : b :: x : y$, by multiplying each term by itself, we shall have, according to the Theorem, $a^2 : b^2 :: x^2 : y^2$; and multiplying these by the given terms, we shall have $a^3 : b^3 :: x^3 : y^3$, &c.

THEOREM XV.

(166.) For *any factors* in an antecedent and its consequent, or the two antecedents, or the two consequents, in a Proportion, may be substituted any other quantities which have the *same ratio to each other*

Let $a : b :: nx : py$,
and $n : p :: r : s$;
then will $a : b :: rx : sy$.

From the two given proportions, we find
$$an : bp :: nrx : psy, \quad (164).$$
Dividing the antecedents in this proportion by n, and the consequents by p,
$$a : b :: rx : sy, \quad (158).$$

This proves the *first affirmation* in the Theorem. By interchanging the *extremes* in the two given proportions, and afterwards the *means*, (156), the other two affirmations in the theorem may be demonstrated.

Therefore, for any factors in an antecedent and its consequent, &c.

GENERAL SOLUTIONS OF PROBLEMS.—FORMULAS. APPLICATIONS OF PROPORTION, &C.

(167.) In the *general solution* of a Problem, all the quantities are represented by *letters;* and the *unknown* being thus found in terms of *all the known* quantities, the result discloses a *rule* for the numerical computation in any given case of the problem.

EXAMPLE.

To find two numbers whose sum shall be s, and difference d.

Let x represent the *greater*, and y the *less* number.

From the conditions of the problem, we have

$$x+y=s,$$

$$\text{and } x-y=d.$$

By adding the second equation to the first; and also subtracting the second from the first, we have

$$2x=s+d, \text{ and } 2y=s-d,$$

which give

$$x=\frac{s+d}{2}, \text{ and } y=\frac{s-d}{2}.$$

From these general values of x and y, we learn that the *greater* of two numbers is equal to $\frac{1}{2}$ of (the *sum* + the *difference*), and that the *less* is equal to $\frac{1}{2}$ of (the *sum* − the *difference*), of the two numbers; hence the following *rule:*

(168.) *To find two numbers from their sum and difference,*—Add the difference to the sum, and divide by 2, for the *greater* of the two numbers; subtract the difference from the sum, and divide by 2, for the *less* number.

For example, if the sum of two numbers be 500, and their difference 146,

$$\text{The greater number is } \frac{500+146}{2}=323;$$

$$\text{and the less number is } \frac{500-146}{2}=177.$$

(169.) An algebraic *Formula* is an equation between the *symbols* of certain quantities—resulting from the general solution of a Problem, or the investigation of some general principle.

Thus the equations which express the *values* of x and y in the preceding Example, are *formulas* for finding two numbers from the *sum* and *difference* of the numbers.

PROBLEMS

In Proportion, Percentage, Interest, &c

(170.) A Proportion occurring in the solution of a Problem, may be converted into an Equation by putting the *product of the two extremes* equal to *the product of the two means*, (149).

EXAMPLE.

To divide \$1000 between three persons, in the proportions of 2, 3, and 5; that is, so that A's share shall be to B's as 2 to 3, and B's to C's as 3 to 5.

Let x represent A's share; y, B's share; and z, C's share. Then, by the conditions of the problem, we have

$$x+y+z=1000;$$
$$x : y :: 2 : 3;$$
$$\text{and } y : z :: 3 : 5.$$

By converting the two Proportions into Equations, we find

$$3x=2y, \text{ and } 5y=3z.$$

We have now *three equations* from which to find the values of the *three unknown quantities*, (125).

The solution of the Problem may also be effected with *one unknown quantity*, by finding *fourth proportionals* for the shares of B and C.

Let x represent A's share;

$$\text{then } 2 : 3 :: x : \frac{3x}{2}; \text{ and } 2 : 5 :: x : \frac{5x}{2}, \quad (150).$$

Hence B's share is represented by $\frac{3x}{2}$, and C's by $\frac{5x}{2}$; and the equation of the problem is

$$x+\frac{3x}{2}+\frac{5x}{2}=\$1000.$$

Ans. The several shares are \$200, \$300, \$500

The *general* Problem of which the preceding is a particular case, may be stated thus;—To divide the sum s between three persons in the proportions of a, b, and c.

The *Formulas* which would be found for the several shares, are

$$\frac{as}{a+b+c}; \quad \frac{bs}{a+b+c}; \quad \frac{cs}{a+b+c}.$$

These Formulas translated into arithmetical language, would furnish a Proposition, or a Rule, which might be applied to any given case of the general problem.

It may be here remarked that algebraic Formulas express general principles and methods of solution with the utmost distinctness and brevity,—but that it is not always possible to translate them into concise and perspicuous phraseology.

EXERCISES.

1. Divide \$950 between two persons so that their shares shall be to each other as 3 to 5.

This problem might be solved without employing *proportion*, by observing that the first share will be $\frac{3}{5}$ of the second.

Ans. \$356$\frac{1}{4}$; \$593$\frac{3}{4}$.

2. Find the Formulas for dividing any given sum s between two persons so that the shares shall be to each other as any two numbers a and b.

Ans. $\frac{as}{a+b}$ and $\frac{bs}{a+b}$.

3. Divide the sum of \$3000 between A, B, and C, in the proportions of 1, 2, and 3. *Ans.* \$500; \$1000; \$1500.

4. Divide the sum of \$7600 between three persons, in the proportions of $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{5}$. *Ans.* \$4000; \$2000; \$1600.

5. A bankrupt is indebted to A \$400, and to B \$700. He is able to pay to both \$900; what sum should each of the two creditors receive?

The \$900 should be divided in the proportion of 400 and 700, or, of 4 and 7, (158). *Ans.* \$327$\frac{3}{11}$; \$572$\frac{8}{11}$.

6. Three persons engaged in a speculation towards which they contributed, respectively, \$300, \$400, and \$500. The profit amounted to \$550; what are the respective shares of profit?

Ans. \$137$\frac{1}{2}$; \$183$\frac{1}{3}$; \$229$\frac{1}{6}$.

7. A, B, and C in a joint mercantile adventure lost \$742. A's part of the capital employed was to B's as 4 to 3, and B's was to C's as 5 to 6; what amount of loss should be borne by each?

Ans. \$280; \$210; \$252.

8. Four persons rented a pasture, in which the first kept 8 oxen, the second 6, the third 10, and the fourth 12. The sum paid was \$40; what amount should have been paid by each person?

Ans. \$8$\frac{8}{9}$; \$6$\frac{2}{3}$; \$11$\frac{1}{9}$; \$13$\frac{1}{3}$.

9. A testator bequeathed his estate, amounting to \$7830, to his three children, in such a manner that the share of the first was to that of the second as 2$\frac{1}{2}$ to 2, and the share of the second to that of the third as 3$\frac{1}{2}$ to 3. What were the shares?

Ans. \$3150; \$2520; \$2160.

10. A, B, C, and D together have \$3000; A's part is to B's as 2 to 3, B and C together have \$1500, and C's part is to D's as 3 to 4 What is the sum possessed by each person?

Ans. \$500; \$750; \$750; \$1000

11. Three persons contributed funds in a joint speculation as follows: A \$200 for 5 months, B \$400 for 3 months, and C \$500 for 4 months. The profit amounted to \$600; what are the several shares of profit?

Each Dollar contributed produced a Profit *proportional to the Time* it was in the business. Each person's share of profit is therefore proportional to his amount of *capital* × its *time;* in other words, the respective shares are to each other in the *compound ratio* of *capital* and *time,* (131).

Hence, A's share of profit is to B's as 200×5 to 400×3;
and A's is to C's as 200×5 to 500×4.

Dividing the antecedent and consequent by 100, these ratios become
2×5 to 4×3 and 2×5 to 5×4, (158).

By still further reductions, on the same principle, the ratios become
5 to 6, and 1 to 2.

Ans. \$142$\frac{6}{7}$; \$171$\frac{3}{7}$; \$285$\frac{5}{7}$.

12. Two persons rented a pasture for \$43. The first put into it 100 sheep for 15 days, and the second 120 sheep for 9 days; what amount of rent should be paid by each person?

Ans. \$25 and \$18.

13. A, B, and C trade together; A ventures \$1000 for 5 months, B \$1200 for 4 months, and C \$800 for 7 months. The profits of the partnership amount to \$2310; what share of profit should be assigned to each? *Ans.* \$750; \$720; \$840.

14. An estate consisting of 1000 acres of land is to be divided between three persons, so that the first share shall be to the second as 2 to 3, and the first to the third as 1 to 2. What are the shares?

Ans. 222$\frac{2}{9}$; 333$\frac{1}{3}$; 444$\frac{4}{9}$, acres.

15. Two men contracted to do a certain work for \$5000. In accomplishing the work, the first employed 100 laborers for 50 days; and the second 125 laborers for 60 days;—to what shares of the stipulated sum are the two men respectively entitled?

Ans. \$2000; and \$3000.

16. A gentleman bequeathed \$18000 to his widow and his three sons, in the proportions of 2, 2$\frac{1}{2}$, 3, and 3$\frac{1}{2}$, respectively. His widow dying before the division was effected, the whole is to be divided proportionably among the three sons. What are their several shares?

Ans. \$5000; \$6000; \$7000.

17. Find the Formulas for dividing, between two partners, the profits s of a joint adventure, in which the first had the Capital a for the Time b, and the second the Capital c for the Time d.

Ans. $\dfrac{abs}{ab+cd}$ and $\dfrac{cds}{ab+cd}$.

Problems in Percentage.

(171.) *Percentage* is an allowance at a certain rate per *hundred;* and this rate is called the *rate per cent.*, from the Latin *centum*, which means a *hundred.*

The *ratio* of percentage is the *rate per cent.* ÷ 100, and is therefore equal to the *rate per unit.*

The *basis* of percentage is the sum or number on which an amount of percentage is computed.

From these definitions it follows, that

(172.) The *basis* of percentage × the *ratio* of percentage produces the *amount* of percentage; and, conversely, that the *amount* ÷ the *ratio* produces the *basis* of percentage.

☞ The Student may be required to write the *Rules* of Percentage which are indicated by the Formulas among the following problems.

18. A merchant finds that his capital, which is now $12000, has increased in one year at the rate of 20 per cent.; what was his capital at the beginning of the year?

Let x represent his capital at the beginning of the year;
then $100 : 120 :: x : 12000$. *Ans.* $10000.

19. What is the Formula for finding a sum of money which, increased at the rate of r per cent., shall amount to the sum a.

Ans. $\dfrac{100a}{100+r}$.

20. An agent receives $500 to be laid out in merchandise, after deducting his commission of $1\frac{1}{2}$ per cent. on the amount of the purchase. What will be the amount of the purchase?

Ans. $492.61'.

21. A merchant obtains an insurance at 2 per cent. on a stock of goods valued at $7500, which includes this amount and the *premium* for the insurance. What is the sum insured? *Ans.* $7653.06'.

22. What is the Formula for finding a sum of money which, diminished at the rate of r per cent., shall leave the sum a?

Ans. $\dfrac{100a}{100-r}$.

23. The profits of a manufacturing company this year amount to $3096, which is $3\frac{1}{4}$ per cent. less than their profits last year. What was the amount of profits last year? *Ans.* $3200.

24. What must be the percentum of profit at which a quantity of merchandise, bought for $3750, must be sold, that the whole amount of profit shall be $1500? *Ans.* 40 per cent.

25. What is the Formula for finding the *rate per cent.* at which the sum s must be increased to produce the sum a?

Ans. $\frac{100(a-s)}{s}$.

26. A quantity of silk was purchased for $220, and, on account of its having become damaged, was sold for $176. What was the percentum of the loss sustained? *Ans.* 20 per cent.

27. What is the Formula for finding the *rate per cent.* at which the sum s must be diminished to leave the sum a?

Ans. $\frac{100(s-a)}{s}$.

28. What amount of stock in an Insurance Office, at a discount of 5 per cent., could be purchased for $3800? *Ans.* $4000.

29. A merchant finds that his capital, which is now $4350, has decreased in one year at the rate of 12½ per cent. What was his capital at the beginning of the year? *Ans.* $4971.42'.

30. What amount of stock in a manufacturing establishment, at an advance of 6½ per cent., could be purchased for $1200?

Ans. $1126.76'.

31. A quantity of damaged cloth was sold for $250,—which was at a loss of 16⅔ per cent. For what sum was the cloth purchased?

Ans. $300.

32. The population of a city increased from 7850 to 11775 inhabitants, in one year. What was the percentum of increase during the year? *Ans.* 50 per cent.

33. An agent receives $2030 to invest in merchandise—himself to retain a commission of 1½ per cent. on the amount of the purchase. What is the sum to be invested? *Ans.* $2000.

34. A merchant wishes to effect an insurance on a stock of goods, amounting to $3573, which shall cover both the value of the goods and the premium of insurance. What is the sum to be insured, allowing the rate to be ¾ per cent? *Ans.* $3600.

35. What amount of stock in a Savings Bank, at an advance of 5 per cent., could be purchased for $4200? and what amount in another, at a discount of 5 per cent., could be purchased for $1995?

Ans. $4000, and $2100.

Problems in Interest, &c.

(173.) *Interest* is the price or premium paid for the *use of money*, and is reckoned at a certain percentum, *annually*, on the sum for which it is paid.

The *Principal* is the sum for which Interest is paid—the *Amount* is the sum of the Principal and Interest.

From the principles of Percentage, (172), it is evident that,

(174.) The Principal × the *ratio* of percentage produces the *interest for one year*. For the ratio of percentage, in this case, is the interest of $1 for one year.

☞ The Student may be required to write the *Rules* of Interest which are indicated by the Formulas among the following problems.

36. What Principal would amount to $1000 in 5 years, allowing the rate of interest to be 6 per cent?

Let x represent the Principal required;

then $x \times \frac{\cdot 6}{100}$ or $\frac{6x}{100}$ is the Interest for one year;

and by adding 5 years' interest to the Principal,

we have $x + \frac{30x}{100} = \$1000$.

Ans. $\$769\frac{3}{13}$.

37. What is the Formula for finding the Principal which, at interest at r per cent., would amount to the sum a in t years?

Ans. $\frac{100a}{100+rt}$.

38. What Principal would amount to $2500, in 10 years, allowing the rate of interest to be 7 per cent.? *Ans.* $1470.588'.

39. At what Rate per cent. must $1000 be put on interest, to amount to $1150 in 2 years and 6 months? *Ans.* 6 per cent.

40. What is the Formula for finding the Rate per cent. of interest at which the sum s would amount to the sum a in t years?

Ans. $\frac{100(a-s)}{st}$.

41. In how many years would $6000 amount to $7470, allowing the rate of interest to be 7 per cent? *Ans.* $3\frac{1}{2}$ years.

42. What is the Formula for finding the Time in which the sum s would amount to the sum a, if the interest be at r per cent?

Ans. $\frac{100(a-s)}{sr}$.

43. What Principal would produce as much interest in $3\frac{1}{2}$ years, as $500 would in 4 years, the rate of interest in both cases being 6 per cent? *Ans.* $$571\frac{3}{7}$.

44. At what Rate per cent. of interest would $525 produce the same amount of interest in 5 years, that $700 would produce at 5 per cent. in 3 years? *Ans.* 4 per cent.

45. A person who possessed a capital of $70000, put the greater part of it at interest at 5 per cent., and the other part at 4 per cent. The interest on the whole was $3250 per annum; required the two parts. *Ans.* $45000, and $25000.

46. The sum of $200 is to be applied in part towards the payment of a debt of $300, and in part to paying the Interest, at 6 per cent., *in advance*, for 12 months, on the remainder of the debt? What is the amount of the payment that can be made on the debt?

Let x represent the payment;

then $(300-x)\times\frac{6}{100}$ is the Interest on the remainder of the debt; and we have therefore the Equation,

$$x+(300-x)\times\tfrac{6}{100}=200.$$

Ans. $193.61+.

47. A is indebted to B $1000, and is able to raise but $600. With this sum A proposes to pay a part of the debt, and the Interest, at 8 per cent., in advance, on his Note at 2 years for the remainder. For what sum should the note be drawn? *Ans.* $476.19'.

48. Find the Formulas for dividing the sum s into two parts, one of which is to be applied towards the payment of a debt of n dollars, and the other to paying the interest, in advance, on the remainder of the debt, for t years, at r per cent. per annum.

Ans. $\frac{100s-nrt}{100-rt}$, and $\frac{rt(n-s)}{100-rt}$.

What would be the RULE for finding the amount of *payment* that could be made on the debt?

CHAPTER VII.

ARITHMETICAL, HARMONICAL, AND GEOMETRICAL PROGRESSION.

ARITHMETICAL PROGRESSION.

(175.) An ARITHMETICAL PROGRESSION is a series of quantities which continually increase or decrease by a *common difference.*

Thus 1, 3, 5, 7, 9, is a Progression in which the quantities increase by the continual *addition* of the common difference 2.

And 15, 12, 9, 6, 3, is a progression in which the quantities decrease by the continual *subtraction* of the common difference 3.

The *first* and *last terms* of the Progression are called the two *extremes*, and all the intermediate terms the *means.*

The theory of Arithmetical Progression is contained in the following propositions.

The Last Term.

(176.) The *last term* of an *increasing* Arithmetical Progression, is equal to the first term + the product of the common difference × the number of terms *less one;* and in a *decreasing* Progression it is equal to the first term — the same product.

Let a be the first term, and d the common difference; then in an increasing progression the series will be,

$$a,\quad a+d,\quad a+2d,\quad a+3d,\quad a+4d,\quad \&c.;$$

and in a decreasing progression the series will be,

$$a,\quad a-d,\quad a-2d,\quad a-3d,\quad a-4d,\ \&c.$$

In these series the *fifth* or *last* term $a\pm 4d$, a plus or minus $4d$, is the first term a plus or minus 4 *times* the common difference d. And the proposition is evidently true for any number of terms.

(177.) *Cor.* The *common difference* of the terms in an Arithmetical Progression, is equal to the difference between the *two extremes* ÷ the number of terms *less one.*

The Sum of the Two Extremes.

(178.) The *sum of the two extremes* in an Arithmetical Progression, is equal to the sum of any two terms *equidistant from them*, or to twice the middle term when the number of terms is *odd*.

Let a be the first term, and d the common difference; then in an increasing progression the series will be,

$$a, \quad a+d, \quad a+2d, \quad a+3d, \quad a+4d, \quad \&c.$$

Of these five terms the sum of the *first* and the *last*, is

$$a+(a+4d)=2a+4d.$$

The sum of the *second* and the *fourth*, which are equidistant from the extremes, is $(a+d)+(a+3d)$, also $=2a+4d$.

We see moreover, that the sum of the two extremes is equal to *twice the middle term*, $a+2d$.

In like manner the proposition will be found true for any number of terms; as also when the Progression is a *decreasing* one.

(179.) *Cor.* An *arithmetical mean* between two given terms, is equal to *half the sum* of those terms.

For the sum of the two given terms, considered as the two extremes of an Arithmetical Progression, is equal to twice the mean or mi[illegible] term.

The Sum of all the Terms.

(180.) The *sum of all the terms* of an Arithmetical Progression, is equal to half the sum of the two extremes × the number of terms.

To prove this proposition we add the several terms of an Arithmetical Progression to those of the same progression *reversed;* thus

$$\begin{array}{cccc}
a, & a+d, & a+2d, & a+3d, \\
\underline{a+3d} & \underline{a+2d} & \underline{a+d} & \underline{a} \\
2a+3d, & 2a+3d, & 2a+3d, & 2a+3d,
\end{array}$$

The sum $(2a+3d)+(2a+3d)$ &c. of the *two series*, is the sum of the two extremes in either series × the number of terms; hence the sum of either series is equal to *half the sum* of the two extremes × the number of terms. The demonstration will evidently apply to any number of terms.

FORMULAS IN ARITHMETICAL PROGRESSION.

(181.) In an Arithmetical Progression, let a be the first term, d the common difference, n the number of terms, l the last term, and S the sum of all the terms. Then

$$A \; . \; . \; . \; . \; l = a \pm d(n-1) \quad (176);$$

$$B \; . \; . \; . \; . \; S = \tfrac{1}{2}n(a+l) \quad (180).$$

The sign $+$ is to be prefixed to $d(n-1)$ when the progression is an increasing one, and $-$ when decreasing.

In these two Formulas we have the *five quantities*, a, d, n, l, s; hence if any *three* of these quantities be given, the values of the other *two* may be found from the two Equations, (121).

HARMONICAL PROGRESSION.

(182.) An HARMONICAL PROGRESSION is a series of quantities such that, of any *three consecutive terms*, the first : the *third* :: the *difference* between the first and second : the *difference* between the second and third.

Thus the numbers 3, 4, 6, 12, are in harmonical progression,

$$\text{since } 3 : 6 :: 4-3 : 6-4,$$
$$\text{and } 4 : 12 :: 6-4 : 12-6.$$

An Harmonical *Proportion* consists of four terms such that the first *is to* the fourth *as* the difference between the first and second *is to* the difference between the third and fourth.

Thus a, b, c, d, are in Harmonical Proportion, if

$$a : d :: a-b : c-d;$$
$$\text{or } a : d :: b-a : d-c.$$

The numbers 16, 8, 3, 2, are in Harmonical Proportion, since

$$16 : 2 :: 16-8 : 3-2 \; .$$

The *first* and *last* terms of the Progression or proportion are called the two *extremes*, and all the intermediate terms the *means*.

An Harmonical converted into an Arithmetical Progression.

(183.) The *reciprocals* of the terms of an Harmonical progression, are in Arithmetical progression.

Let a, b, c, be three consecutive terms of a decreasing harmonical progression.

$$\text{Then } a : c :: a-b : b-c, \quad (182);$$

Converting this proportion into an equation, we have

$$ab-ac=ac-bc.$$

Dividing each term in this equation by abc, and reducing the several quotients to their *lowest terms*, we find

$$\frac{1}{c}-\frac{1}{b}=\frac{1}{b}-\frac{1}{a}.$$

Transposing the first and the last term of this equation

$$\frac{1}{a}-\frac{1}{b}=\frac{1}{b}-\frac{1}{c}.$$

We thus find that the difference between the reciprocals of a and b, is equal to the difference between the reciprocals of b and c; hence these reciprocals are in Arithmetical Progression, (175).

The numbers 3, 4, 6, 12, form an Harmonical Progression, (182): by taking the *reciprocals* of the several terms we have the Arithmetical Progression

$$\tfrac{1}{3},\ \tfrac{1}{4},\ \tfrac{1}{6},\ \tfrac{1}{12};$$

in which the common *difference* of the terms is $\frac{1}{12}$.

(184.) An *harmonical mean* between two given terms, is equal to *twice their product* divided by their *sum*.

From the first of the preceding equations, namely,

$$ab-ac=ac-bc,$$

$$\text{we shall find } b=\frac{2ac}{a+c};$$

and b is the harmonical mean between a and c.

The harmonical mean between 3 and 6, is

$$\frac{3\times6\times2}{3+6}=\frac{36}{9}=4.$$

GEOMETRICAL PROGRESSION.

(185.) A GEOMETRICAL PROGRESSION is a series of quantities in which each succeeding term has the *same ratio* to the term which immediately precedes it.

Thus 1, 2, 4, 8, 16, is an increasing Progression in which each succeeding term is *double* the one which immediately precedes it; that is, the *ratio of the progression* is 2.

And 27, 9, 3, 1, $\frac{1}{3}$, is a decreasing progression in which each succeeding term is *one-third* of the one which immediately precedes it; and the ratio of the progression is consequently $\frac{1}{3}$.

Hence the *successive terms* of a Geometrical Progression consist of the *first term multiplied* continually into the *ratio*, that is, multiplied into the successive *powers of the ratio.*

The theory of Geometrical Progression is contained in the following propositions.

The Last Term.

(186.) The *last term* of a Geometrical Progression, is equal to the first term × that *power of the ratio* which is expressed by the number of terms *less one.*

Let a be the first term, and r the *ratio* of the progression; then, multiplying a continually into r, the series will be

$$a, \quad ar, \quad ar^2, \quad ar^3, \quad ar^4, \quad \&c.$$

Since the ratio r begins in the second term, with *exponent* 1, its exponent in the last term will always be *one less* than the number of terms; hence the last term consists of the first × into that power of r which is expressed by the number of terms *minus* 1.

(187.) *Cor.* The *last term* of a Geometrical Progression ÷ the first term, gives that *power of the ratio* which is expressed by the number of terms *less one.*

Thus $ar^4 \div a = r^4$; the number of terms being *five.*

Product of the two Extremes.

(188.) The *product of the two extremes* in a Geometrical Progression, is equal to the product of any two terms *equidistant from them*, or to the *square of the middle term* when the number of terms is *odd*.

Let a be the first term, and r the ratio of the progression; then, multiplying a into the successive *powers* of r, the series is

$$a, ar, ar^2, ar^3, ar^4, \&c.$$

Of these five terms the product of the *first* and the *last*, is

$$a \times ar^4 = a^2r^4.$$

The product of the *second* and the *fourth*, which are equidistant from the extremes, is $ar \times ar^3$, also $= a^2r^4$.

We perceive moreover that the product of the two extremes is equal to the *square of the middle term* ar^2.

In like manner the proposition will be found true for any number of terms.

(189.) *Cor.* A *geometrical mean*, or a mean proportional, between two given terms, is equal to the *square root of the product* of those terms.

For the product of the two given terms, considered as the two extremes of a Geometrical Progression, is equal to the square of the mean or middle term.

The Sum of all the Terms.

(190.) The *sum of all the terms* of a Geometrical Progression, is equal to the difference between the *first term* and the *product* of the *last term* × the *ratio*, ÷ the difference between the *ratio* and a *unit*.

Let S represent the sum of the terms, and we shall have

$$S = a + ar + ar^2 + ar^3 + ar^4, \&c.$$

Multiplying both sides of this equation by the ratio r,

$$Sr = ar + ar^2 + ar^3 + ar^4 + ar^5.$$

Subtracting the first equation from the second, we find

$$Sr - S = ar^5 - a;$$

which gives $S = \dfrac{ar^5 - a}{r - 1}$.

In the numerator of this value of S, observe that ar^5 is the last term ar^4 of the progression × the ratio r. The demonstration will evidently apply to any number of terms.

(191.) The sum of an *infinite number of terms* in a decreasing Geometrical Progression, is equal to the *first term* divided by the difference between the *ratio* and a *unit.*

In a decreasing progression the terms continually diminish in a *constant ratio;* and if the number of terms be *infinite,* the last term will be 0. The last term × the *ratio* will then be 0, and the expression for the sum of the terms, found above, will become

$$S=\frac{a}{1-r}.$$

The divisor in this case is $1-r$, because the ratio of the progression being a *proper fraction,* is less than a *unit.*

FORMULAS IN GEOMETRICAL PROGRESSION.

(192.) In a Geometrical Progression, let a be the first term, r the ratio, n the number of terms, l the last term, and s the sum of all the terms. Then

$$\text{C} \;.\;.\;.\;.\; l=ar^{n-1}, \quad (186),$$

$$\text{D} \;.\;.\;.\;.\; S=\frac{lr-a}{r-1}, \quad (190).$$

When the progression is a decreasing one, and the number of terms is infinite,

$$\text{E} \;.\;.\;.\;.\; S=\frac{a}{1-r}, \quad (191).$$

In the Formulas C and D we have the five quantities, a, r, n, l, s; hence if any *three* of these quantities be given, the values of the other *two* may be found from the two Equations, (121).

The principles which have been established in this Chapter may be applied to the solution of the following

Problems in Progressions.

1. The first term of an increasing Arithmetical progression is 3, the common difference of the terms is 2, and the number of terms 20. What is the *last term?* and the sum of all the terms?

Ans. 41, and 440.

2. The first term of a decreasing Arithmetical progression is 100, the common difference of the terms is 3, and the number of terms 34. What is the last term? and the sum of all the terms?

Ans. 1, and 1717.

3. What is the sum of the numbers 1, 2, 3, 4, 5, &c., continued to 1000 terms? *Ans.* 500500.

4. What is the *common difference* of the terms in an Arithmetical progression whose first term is 10, last term 150, and number of terms 21? *Ans.* 7.

5. If the *third* term of an Arithmetical progression be 40, and the *fifth* term 70, what will the *fourth* term be? *Ans.* 55.

6. If the first term of an Arithmetical progression be 5, and the fifth term 30, what will the *second, third,* and *fourth* terms be?

Find the *common difference,* (177), and thence the three intermediate terms. *Ans.* 11¼; 17½; 23¾.

7. If the fourth term of an Arithmetical progression be 37, and the eighth term 60, what are the intermediate terms?

Ans. 42¾; 48½, 54¼.

8. What is the sum of 25 terms of an increasing Arithmetical progression in which the first term is ½, and the common difference of the terms also ½? (176). *Ans.* 162½.

9. The first term of an increasing Arithmetical progression, is 1, and the number of terms 23. What must be the *common difference,* that the sum of all the terms may be 100?

Let x represent the common difference;

then $1+22x$ is the last term, (176);

and $\frac{2+22x}{2}\times 23$ is the sum of the terms, (180).

Hence an Equation may be formed from which the value of x will be found.—Or we might substitute the numbers 1, 23, and 100, for a, n, and s in Formulas A and B, (181), and find the value of d, as one of the *two unknown quantities.* *Ans.* $\frac{7}{23}$.

10. If the first term of a decreasing Arithmetical progression is 100, and the number of terms 21, what must the common difference be, that the sum of the series may be 1260? *Ans.* 4.

11. A and B start together, and travel in the same direction; A goes 40 miles per day; B goes 20 miles the first day, and increases his rate of travel ¾ of a mile per day. How far will they be apart at the end of 40 days? *Ans.* 215 miles.

12. One Hundred stones being placed on the ground in a straight line, at the distance of 2 yards from each other; how far will a person travel who shall bring them, one by one, to a basket which is placed 2 yards from the first stone? *Ans.* 11 miles 840 yards.

13. Find the *third term* of an Harmonical progression whose first and second terms are 12 and 15 respectively.

If x represent the third term, we shall have

$12 : x :: 15-12 : x-15.$ *Ans.* 20.

14. What is the first term of an Harmonical progression whose second and third terms are 30 and 20 respectively? *Ans.* 60.

15. What is the *fourth* term of an Harmonical *proportion* whose first, second, and third terms are 2, 3, and 8 respectively?

Ans. 16.

16. If the first and third terms of an Harmonical progression be 25 and 40 respectively, what will the second term be? *Ans.* $30\frac{10}{13}$.

17. The first and fourth terms of an Harmonical progression, are 10 and 20 respectively. What are the two intermediate terms?

This problem may be solved by finding two *arithmetical means* between $\frac{1}{10}$ and $\frac{1}{20}$, and then taking the *reciprocals* of the terms thus found, (183). *Ans.* 12, and 15.

18. The fifth and eighth terms of an Harmonical progression are 20 and 40 respectively. What are the two intermediate terms?

Ans. 24, and 30.

19. The first term of a Geometrical progression is 2, the ratio of the progression is 3, and the number of terms 4. What is the *last term?* and the sum of all the terms? *Ans.* 54, and 80.

20. The first term of a Geometrical progression is $\frac{1}{2}$, the ratio of the progression is $\frac{1}{3}$, and the number of terms 4. What is the last term? and the sum of all the terms? *Ans.* $\frac{1}{54}$, and $\frac{20}{27}$.

21. What is the sum of an infinite number of terms in the Geometrical progression whose first term is 100, and ratio $\frac{1}{4}$? *Ans.* $133\frac{1}{3}$.

22. What is the sum of an infinite number of terms in the Geometrical progression whose first term is 300, and ratio $\frac{1}{3}$? *Ans.* 450.

23. If the first and third terms of a Geometrical progression are 8 and 72 respectively, what is the *second* term?

The second term is equal to the *square root* of 8×72, (189).

Or, considering the third as the *last term* of the progression,

$72\div8=9$ is the *square* of *the ratio*, (187);

then 3 is the *ratio of the progression;* and the second term is now readily obtained. *Ans.* 24.

24. If the third and fifth terms of a Geometrical progression be 75 and 300 respectively, what will the fourth term be? *Ans.* 150

25. If the first and fourth terms of a Geometrical progression are 3 and 24 respectively, what are the two intermediate terms?
Ans. 6 and 12.

26. If the seventh and tenth terms of a Geometrical progression are 6 and 750 respectively, what are the intermediate terms?
Ans. 30 and 150.

27. What is the sum of an infinite number of terms in the series 1, $\frac{1}{2}$, $\frac{1}{4}$, &c., in which the ratio of the progression is evidently $\frac{1}{2}$?
Ans. 2.

28. If a body move forever at the rate of 2000 feet the first second. 1000 the second, 500 the third, and so on, what is the utmost distance it can reach? *Ans.* 4000.

29. If 10 yards of cloth be sold at the rate of $1 for the first yard, $2 for the second, $4 for the third, and so on, what would be the price of the last yard? and what would the whole amount to?
Ans. $512, and $1023.

30. If 13 acres of land were purchased at the rate of $2 for the first acre, $6 for the second, $18 for the third, and so on, what would the last acre amount to? *Ans.* $1062882.

31. Allowing the interest of a sum of money to be $500 the first year, $400 the second, $320 the third, and so on, forever, what would be the whole amount of interest? *Ans.* $2500.

32. Two bodies move at the same time, from the same point, in opposite directions. One goes 2 miles the first hour, 4 the second, 6 the third, and so on; the other goes 2 miles the first hour, 4 the second, 8 the third, &c.; how far will they be apart at the end of 12 hours? *Ans.* 8346 miles.

33. A and B set out at the same time to meet each other. A travels 3, 4, 5, &c. miles on successive days, and B 3, 4$\frac{1}{2}$, 6$\frac{3}{4}$, &c. miles on successive days. They meet in 10 days; what is the distance between the two places from which they traveled?
Ans. 414$\frac{507}{512}$ miles.

CHAPTER VIII.

PERMUTATIONS AND COMBINATIONS.—INVOLUTION.—BINOMIAL THEOREM.—EVOLUTION.

PERMUTATIONS.

(193.) PERMUTATIONS are the different *orders of succession* in which a given number of things may be taken—either the *whole number together*, or the whole number taken *two* and *two*, or *three* and *three*, &c.

Thus the different Permutations of the three letters a, b, and c, when *all are taken together*, are

$$abc, \quad acb, \quad bac, \quad cab, \quad bca, \quad cba.$$

And the different Permutations of the same letters when taken *two* and *two*, are $ab, \; ba, \; ac, \; ca, \; bc, \; cb.$

Number of Permutations.

(194.) If n represent a given number of things, the number of *permutations* that can be formed of them, will be equal to

$$n(n-1)(n-2)(n-3)(n-4), \text{ and so on,}$$

until the number of *factors* multiplied together is equal to the number of things taken in *each permutation*.

To demonstrate this proposition,—suppose that we have n letters, a, b, c, d, &c., to be subjected to Permutations.

If we reserve one of the letters, as a, there will remain $n-1$ letters; and by writing a before each of the remaining letters, we have

$n-1$ permutations of n letters, taken *two* and *two*, in which a stands first.

In like manner we should find $n-1$ permutations of n letters, taken *two* and *two*, in which b stands first; and so for each of the n letters. Hence we shall have

$n(n-1)$ *permutations* of n letters taken *two* and *two*.

Suppose now that the n letters are to be taken *three* and *three*.

By reserving a, and proceeding with $n-1$ letters as before, we should find $(n-1)(n-2)$ permutations of $n-1$ letters taken *two* and *two;* and by writing a before each of these permutations, we have

$(n-1)(n-2)$ permutations of n letters, taken *three* and *three*, in which a stands first.

We should find the same number of permutations of n letters, taken *three* and *three*, in which b stands first; and so for each of the n letters. Hence we shall have

$n(n-1)(n-2)$ *permutations* of n letters taken *three* and *three*.

Suppose now that the n letters are to be taken *four* and *four*.

By reserving a, and pursuing the operation in the same manner as before, we should find

$n(n-1)(n-2)(n-3)$ *permutations* of n letters taken *four* and *four*.

Thus the demonstration proceeds; the number of *factors* multiplied together being found always equal to the number of letters taken in *each permutation*.

As an Example of the application of the principle above demonstrated,—suppose it were required to determine the number of Permutations, or different *orders of succession*, that could be formed in a class composed of *six* pupils, by taking the *whole number* in each permutation.

Since the *six* pupils are to be taken in each Permutation, the number of *factors* to be employed is *six*; hence the number of permutations is

$$6\times5\times4\times3\times2\times1=720.$$

If the whole *six* were to be subjected to Permutations by taking *five* at a time, the number of permutations would be

$$6\times5\times4\times3\times2=720.$$

If the whole *six* were to be subjected to Permutations by taking *four* at a time, the number of permutations would be

$$6\times5\times4\times3=360.$$

☞ It will be observed above, that the number of Permutations will be the same, whether the whole number of things, or one less than the whole number, be taken in each permutation.

COMBINATIONS.

(195.) COMBINATIONS are the different *collections* which may be formed out of a given number of things, by taking the same number in each collection—without regard to the order of succession.

Thus the different Combinations which may be formed out of the three letters a, b, and c, by taking two at a time, are

$$ab,\quad bc,\quad ac.$$

Observe that ab and ba are not different *combinations*, but different *permutations*, of the letters a and b.

In Permutations we have regard to the *order of succession*, and may therefore have two permutations of *two things*. In Combinations we do not consider the order of succession; so that the combination of two or more things is the same, in whatever order they are taken.

Number of Combinations.

(196.) If n represent a given number of things, the number of *combinations* that can be formed out of them, will be equal to

$$\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}, \text{ and so on,}$$

until the number of *factors* in the dividend, and also in the divisor, is equal to the number of things taken in *each combination*.

To demonstrate this proposition, we observe that the *numerator* in the preceding expression, is the number of *permutations* of n things taken *four* and *four*, (194).

On the same principle, the *denominator* $4\times3\times2\times1$ is the number of permutations of *four* things taken *all together*.

Now since there can be but *one combination* of 4 things taken *all together*, the number of *permutations* of n things taken *four* and *four* is 1.2.3.4 *times*, that is, 24 *times*, the number of *combinations*.

Hence the number of Combinations of n things taken *four* and *four*, is $\frac{1}{24}$ of the number of Permutations; and will therefore be found by dividing the numerator by the denominator.

This mode of demonstration may evidently be applied whatever be the number of things taken in each combination.

PROBLEMS

In Permutations and Combinations.

1. In how many different ways might a company of 10 persons seat themselves around a table? (194). *Ans.* 3628800.

2. How many different numbers might be expressed by the 10 numeral figures, if 5 figures be used in each number? *Ans.* 30240.

3. In how many different ways may the names of the 12 months of the year be arranged one after another? *Ans.* 479001600.

4. How many different permutations of 8 men could be formed out of a company consisting of 15 men? *Ans.* 259459200.

5. In how many different ways might the seven prismatic colors, *red*, *orange*, *yellow*, *green*, *blue*, *indigo*, and *violet*, have been arranged in the solar spectrum? *Ans.* 5040.

6. How many different *combinations* of *two colors* could be formed out of the 7 prismatic colors? (196). *Ans.* 21.

7. How many different combinations of 5 letters may be formed out of the 26 letters of the Alphabet? *Ans.* 65780.

8. How many different combinations of 2 *elements* might be formed out of the 56 *elements* described in Chemistry? *Ans.* 1540.

9. In how many different ways might a company of 20 men be arranged, in single file, in a procession?

Ans. 2432902008176640000.

10. A farmer wishes to select a team of 6 horses out of a drove containing 10 horses. How many different choices for the team will he be able to make? *Ans.* 210.

11. In how many different ways might the planets Mercury, Venus, the Earth, Mars, Jupiter, Saturn, Uranus, and Neptune succeed one another in the solar system? *Ans.* 40320.

12. A company of 20 persons engaged to remain together so long as they might be able to combine in different couples in their evening walks. What time will be required to fulfil the engagement?

Ans. 190 days.

13. How many different permutations of 7 letters might be formed out of the 26 letters of the Alphabet? *Ans.* 3315312000.

14. In an exhibition of a Public School, 5 speakers are to be taken from a class of 15 students. How many different selections of the five might be made? and in how many different ways might the 5 succeed one another in the delivery of their speeches?

Ans. 3003 and 120.

15. Out of a Company consisting of 100 soldiers Six are to be taken for a particular service. How many different selections of the 6 might be made? and in how many different ways might the 6 chosen be disposed with regard to the *order of succession?*

Ans. 1192052400; and 720

INVOLUTION.

(197) INVOLUTION consists in raising a given quantity to any required *power*. This may always be effected by multiplying the quantity into itself as many times *less one* as there are units in the *exponent* of the power.

Thus aa is a^2, the second *power* or *square* of a;
aaa is a^3, the third *power* or *cube* of a; and so on.

Observe that one multiplication of a into itself produces the *second* power of a; two multiplications produce the *third* power, and so on; also that the number of times the quantity becomes a *factor* in raising a Power, is equal to the *exponent* of the Power.

(198.) A *higher power* of a quantity may also be found by multiplying together two or more lower powers (of the same quantity) the *sum of whose exponents* is equal to the exponent of the required power.

Thus $a^2 \times a^2$ produces a^4; $a^2 \times a^3$ produces a^5, &c.

Powers of Unity, Monomials, Fractions, &c.

(199.) Every power of *unity* is *unity*, since any number of 1s multiplied together produce only 1; thus $1 \times 1 \times 1$ &c. $= 1$.

(200.) A *monomial* is raised to any required Power by raising its numerical *coefficient* to that power, and multiplying the *exponents* of its other factors by the exponent of the *power*.

Thus to find the *third* power of $4ax^2$, we raise 4 to its *third* power, which is $4 \times 4 \times 4 = 64$, and multiply the exponents of a and x by 3: we thus obtain $64a^3x^6$.

Observe that multiplying the given *exponents* by the exponent of the required Power, is only a brief method of performing the requisite multiplications in the Involution of Monomials.

(201.) A *fraction* is raised to any required Power, by raising its numerator and denominator, separately, to that power.

Thus the *third* power of $\frac{a^2}{2x}$ is $\frac{a^6}{8x^3}$. (98).

A *mixed quantity* may be raised to any required Power by involving the equivalent *improper fraction*.

The Sign to be Prefixed to a Power.

(202.) Every *even power* of a quantity is *positive;* while every *odd* power has the same sign as the *quantity from which it is derived.*

Thus if a be *positive,* all its powers, as aa, aaa, and so on, will evidently be *positive;* but if a be *negative* we shall have

$$-a.-a=a^2\,;\ a^2.-a=-a^3\,;\ -a^3.-a=a^4\,;\ a^4.-a=-a^5, \text{\&c.}$$

from which it is plain that all the *even* powers, as the 2d, 4th, and so on, will be *positive,* while all the *odd* powers will be *negative.*

EXERCISES.

On the Powers of Monomials.

1. Find the square of $3ax^2$. *Ans.* $9a^2x^4$
2. Find the cube of $-2a^2x$. *Ans.* $-8a^6x^3$.
3. Find the square of $-4ax^2$. *Ans.* $16a^2x^4$.
4. Find the cube of $3a^3x$. *Ans.* $27a^9x^3$.
5. Find the square of ab^2c^3. *Ans.* $a^2b^4c^6$.
6. Find the cube of $-a^2x^2y$. *Ans.* $-a^6x^6y^3$.
7. Find the square of $\frac{1}{2}ax^2$. *Ans.* $\frac{1}{4}a^2x^4$.
8. Find the cube of $\frac{1}{2}ay^3$. *Ans.* $\frac{1}{8}a^3y^9$.
9. Find the square of $-\frac{2}{3}ab^n$. *Ans.* $\frac{4}{9}a^2b^{2n}$.
10. Find the cube of $-\frac{2}{3}ax^2$. *Ans.* $-\frac{8}{27}a^3x^6$.
11. Find the 4th power of $2a^n$. *Ans.* $16a^{4n}$.
12. Find the 4th power of $-\frac{1}{2}a^2$. *Ans.* $\frac{1}{16}a^8$.
13. Find the 4th power of $-3x^2$. *Ans.* $81x^8$.
14. Find the 5th power of $\frac{1}{2}x^2$. *Ans.* $\frac{1}{32}x^{10}$.
15. Find the 6th power of $-ax^2$. *Ans.* a^6x^{12}
16. Find the 6th power of $2y^2$. *Ans.* $64y^{12}$
17. Find the 7th power of $-a^2y^n$. *Ans.* $-a^{14}y^{7n}$.
18. Find the 7th power of $\frac{1}{2}y$. *Ans.* $\frac{1}{128}y^7$.
19. Find the 8th power of a^2b^n. *Ans.* $a^{16}b^{8n}$.
20. Find the 8th power of $-\frac{1}{2}x^2$. *Ans.* $\frac{1}{256}x^{16}$.

Powers of Polynomials.

(203.) The Powers of a *binomial*, or of any *polynomial*, may be obtained by successive multiplications of the quantity into itself, (197).

Thus the 2d power or *square* of $a+b$, is

$$(a+b)(a+b)=a^2+2ab+b^2.$$

And the 3d power or *cube* of $a+b$, is

$$(a+b)(a+b)(a+b)=a^3+3a^2b+3ab^2+b^3.$$

The Involution of Binomials, however, and thence of Polynomials, is greatly facilitated by the application of Newton's

Binomial Theorem.

(204.) This Theorem explains a general method of *developing* a Binomial according to any *exponent* with which the Binomial may be affected.

In developing $(a+b)^2$ by the Binomial Theorem we should obtain the 2d *power* or square of $(a+b)$.

In developing $(a+b)^{\frac{1}{2}}$ we should obtain the *square root* of $(a+b)$; and so for other exponents.

Development of $(a\pm b)^n$;

n representing any *exponent*.

1. The *Exponent* of a in the *first term* of the development, is the same as the *exponent of the Binomial*, and *decreases* by the subtraction of 1, continually, in the succeeding terms.

2. The Exponent of b entering as a *factor*, commences with 1 in the *second* term of the development, and *increases* by the addition of 1, continually, in the succeeding terms.

3. The *Coefficient* of the *first term* is 1; that of the *second* term is the same as the *exponent of the Binomial*;—and the coefficient of the second, or of any term, × the exponent of a in that term, and ÷ by the *exponent* of b increased by 1, gives the coefficient of the succeeding term.

4. The *Signs* of the terms in the development of $(a+b)$ will all be $+$; while for $(a-b)$ they will be alternately $+$ and $-$.

From these principles we have the following

(205.) *Formula for the Development of the Binomial* $(a+b)^n$.

$$(a+b)^n=a^n+na^{n-1}b+\frac{n(n-1)}{2}a^{n-2}b^2+\frac{n(n-1)(n-2)}{2\times3}a^{n-3}b^3, \text{ \&c}$$

When the exponent n is a *positive integer*, as 2, 3, or 4, &c., the development will *terminate* at the term in which the exponent of b becomes equal to the exponent of the Binomial.

For the exponent of a in that term will be 0, and 0 will thus become a *factor* in finding the *coefficient* of the next term; hence the next term will be 0, (43).

EXAMPLE I.

To find the 4th *power* of $(a+b)$, by the Binomial Theorem;
that is, to *develop* $(a+b)^4$.

The *literal* factors without the *coefficients*, will be

$$a^4 \quad a^3b \quad a^2b^2 \quad ab^3 \quad b^4$$

By *computing* and introducing the *coefficients*, with the *signs*, we have

$$a^4+4a^3b+6a^2b^2+4ab^3+b^4.$$

The student will readily perceive the application of the preceding principles to the several terms of this Power.

Demonstration of the Binomial Theorem.

The principles which have been given for the development of $(a\pm b)^n$, will be demonstrated under the supposition that the exponent n is a *positive integer*.—A general demonstration would be equally applicable to *negative* or *fractional* exponents; such demonstration is unnecessary here, and is too abstruse for the present stage of our subject.

If we multiply together the *binomial factors*

$$a+b, \quad a+c, \quad a+d, \quad a+e,$$

and decompose the product terms which contain the *lower powers* of a, the final Product may be represented as follows;

$$\begin{array}{ll|ll|ll|ll}
a^4 & +b & a^3 & +bc & a^2 & +bcd & a & +bcde \\
 & +c & & +bd & & +bce & & \\
 & +d & & +be & & +bde & & \\
 & +e & & +cd & & +cde & & \\
 & & & +ce & & & & \\
 & & & +de & & & &
\end{array}$$

In this Product observe that the *coefficients* of the powers of a in the successive terms, are as follows;

The Coefficient of a^4 is 1; the coefficient of a^3 is the sum of the second terms, b, c, &c., of the binomial factors composing the Product; the coefficient of a^2 is the sum of the products bc, bd, &c., of the second terms of the binomial factors *combined* two and two, (196); and the coefficient of a is the sum of the products bcd, bce, &c., of the second terms of the binomial factors *combined* three and three.

Suppose now that c, d, e, &c. are each equal to b, and the number of binomials equal to n. The Product of these factors will be the nth power of $(a+b)$; that is, it will be

the *development* of $(a+b)^n$.

The Exponent of a in the first term of the Product will evidently be n; and, as exemplified in the preceding multiplication, this exponent will decrease by 1, continually, in the succeeding terms.

Hence we shall have

a^n, a^{n-1}, a^{n-2}, &c. in the consecutive terms.

The Coefficient of a^n will evidently be 1. The coefficient of a^{n-1} in the second term, will be n times b.

Hence the second term will be

$$nba^{n-1} \text{ or } na^{n-1}b.$$

The Coefficient of a^{n-2} in the third term, will be b^2 taken *as many times* as there are combinations of *two* letters in n letters, (196).

Hence the third term will be

$$\frac{n(n-1)}{1\times 2}b^2a^{n-2} \text{ or } \frac{n(n-1)}{1\times 2}a^{n-2}b^2.$$

The Coefficient of a^{n-3} in the fourth term, will be b^3 taken as many times as there are combinations of *three* letters in n letters.

Hence the fourth term will be

$$\frac{n(n-1)(n-2)}{1\times 2\times 3}b^3a^{n-3} \text{ or } \frac{n(n-1)(n-2)}{1\times 2\times 3}a^{n-3}b^3.$$

The several terms thus obtained agree with *Formula* (205); and the *law of development* thus indicated, will, in like manner, be found applicable to any number of terms.

With regard to the *Signs*,—it is evident that when a and b are *positive*, all the terms in any power of $(a+b)$ will be *positive*, (42).

When b is *negative*, all the *odd* powers of b will be *negative*, (202); and these negative powers multiplied by the positive powers of a, will cause the 2d, 4th, 6th, &c. terms to become *negative*.

EXAMPLE II.

To find the 5th power of the binomial $a-x$.

The literal factors without the *coefficients* are

$$a^5,\quad a^4x,\quad a^3x^2,\quad a^2x^3,\quad ax^4,\quad x^5.$$

By computing and inserting the coefficients, with the **signs**, we have

$$a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5.$$

EXERCISES

On the Powers of Polynomials.

1. Find the square of $a-x$. *Ans.* $a^2-2ax+x^2$.
2. Find the cube of $a+y$. *Ans.* $a^3+3a^2y+3ay^2+y^3$.
3. Find the cube of $a+2b$.

By applying the principles of the Binomial Theorem, we obtain

$$a^3+3a^2.2b+3a(2b)^2+(2b)^3;$$

which may be developed into *Ans.* $a^3+6a^2b+12ab^2+8b^3$.

4. Find the square of $3a+y$. *Ans.* $9a^2+6ay+y^2$.
5. Find the cube of $2a-3x$. *Ans.* $8a^3-36a^2x+54ax^2-27x^3$.
6. Find the square of $a+x-y$,

By operating on $(x-y)$ as if it were a *monomial*, and applying the Binomial Theorem to $a+(x-y)$, we obtain

$$a^2+2a(x-y)+(x-y)^2;$$

which may be developed into *Ans.* $a^2+2ax-2ay+x^2-2xy+y^2$.

7. Find the square of $a-b+y$. *Ans.* $a^2-2ab+2ay+b^2-2by+y^2$.
8. Find the square of $a-x-y$. *Ans.* $a^2-2ax+x^2-2ay+2xy+y^2$.
9. Find the square of $a+b-2x$.
   Ans. $a^2+2ab+b^2-4ax-4bx+4x^2$.
10. Find the square of a^2+x^3.

This square, according to the Binomial Theorem, may be indicated thus:

$$(a^2)^2+2a^2x^3+(x^3)^2;$$

which may be developed into *Ans.* $a^4+2a^2x^3+x^6$.

11 Find the cube of a^2-y^3. *Ans.* $a^6-3a^4y^3+3a^2y^6-y^9$.

12. Find the cube of $1+2x^2$. *Ans.* $1+6x^2+12x^4+8x^6$.

13 Find the fourth power of $a+x$.
Ans. $a^4+4a^3x+6a^2x^2+4ax^3+x^4$

14. Find the fifth power of $a-y$.
Ans. $a^5-5a^4y+10a^3y^2-10a^2y^3+5ay^4-y^5$

EVOLUTION.

(206.) Evolution consists in extracting any required *root* of a given quantity, regarded as the corresponding power of the root to be found.

Extracting the square root consists in finding a quantity *whose square is equal to a given quantity;* extracting the *cube root* consists in finding a quantity whose *cube* is equal to a given quantity; and so on.

Roots of Unity, Monomials, Fractions, &c.

(207.) Every root of *unity* is *unity*, since the *square*, or cube, &c. of 1 is 1; thus $1\times1=1$; $1\times1\times1=1$; and so on.

In general terms, any *power* or *root* of a unit is a unit.

(208.) Any required Root of a *monomial* will be found by extracting the root of its numerical *coefficient*, and dividing the *exponents* of its literal factors by the *integer* corresponding to the root.

Thus the *square root* of $25\,a^2x$ is $5ax^{\frac{1}{2}}$, found by extracting the square root of 25, and dividing the exponents of a and x by 2.

And the *cube root* of $27a^6x^{\frac{1}{2}}$ is $3a^2x^{\frac{1}{6}}$, found by extracting the cube root of 27, and dividing the exponents of a and x by 3.

The correctness of this method will appear from considering that the Extracting of a Root is the reverse of raising the corresponding Power, (200).

Hence also the propriety of denoting roots by *fractional* exponents.

$a^{\frac{1}{2}}$ denotes the square root of a, because $a^{\frac{1}{2}}\times a^{\frac{1}{2}}=a$.

(209.) Any required Root of a *fraction* will be found by extracting the root of its numerator and denominator, separately.

Thus the square root of $\dfrac{4x^2}{9a^2y^4}$ is $\dfrac{2x}{3ay^2}$, found by extracting the square root of the numerator, and the square root of the denominator.

A Root of a *mixed quantity* would be found by extracting the root of the equivalent *improper fraction.*

(210.) A Root whose exponent is *resolvable into two factors*, may be found by extracting *in succession* the roots denoted by those factors.

The 4th root may be found by extracting the *square root* of the *square* root,—the exponent $\frac{1}{4}$ which denotes the 4th root, being $\frac{1}{2} \times \frac{1}{2}$.

Thus the 4th root of x^4 is the square root of x^2; the 4th root of 81 is the square root of 9; and the 4th root of 10000 is the square root of 100.

The 6th root may be found by extracting the square root of the *cube root*, or the cube root of the square root,—$\frac{1}{6}$ being equal to $\frac{1}{2} \times \frac{1}{3}$.

The correctness of this method of extracting Roots, is evident from considering, that, by raising in succession the *powers* denoted by two or more factors, we shall obtain the power denoted by the *product* of those factors.

Thus the *square* of the *cube* of a is $(a^3)^2 = a^6$, (200), and therefore, conversely, the *square root* of the *cube root* is the *sixth* root

Roots of Powers or Powers of Roots.

(211.) The Numerator of a fractional exponent denotes a *power* of the quantity affected, and the Denominator a *root of that power.*

Thus $a^{\frac{1}{2}}$ denotes the square root of the *first power* of a, or simply the square root of a. In like manner $a^{\frac{2}{3}}$ denotes the *cube root* of the square of a; $a^{\frac{3}{4}}$ denotes the 4th root of the cube of a; and so on. But, observe that

(212.) A Root of any *power* of a quantity, is equal to the same *power* of the same *root* of the quantity.

To illustrate this principle it may be shown that the *cube root* of the 6th *power* of a, is equal to the 6th power of the cube root of a.

The *cube root* of a^6 is a^2 since $a^2.a^2.a^2 = a^6$; and the 6th *power* of $a^{\frac{1}{3}}$ is also a^2, since the *cube* of $a^{\frac{1}{3}}$ is a, and the *square of this cube*, —which gives the 6th *power* of $a^{\frac{1}{3}}$,—is a^2, (200).

To give an application of this principle to *numbers*,—the cube root of the square of 8, is 4, and the square of the cube root of 8 is 4

Equivalent Exponents.

(213.) Two Exponents which are numerically equivalent, are also equivalent *exponentially;* and may therefore be substituted, the *one for the other.*

Thus $a^{\frac{2}{4}}=a^{\frac{1}{2}}$; the *fourth root* of a^2 is equivalent to the *square root* of a. The 4th root of a^2 is $a^{\frac{1}{2}}$, since $a^{\frac{1}{2}}$ raised to the 4th *power* produces a^2.

In like manner $x^{\frac{3}{6}}=x^{\frac{1}{2}}$; $y^{\frac{4}{6}}=y^{\frac{2}{3}}$; $a^{\frac{4}{2}}=a^2$; &c.

To give an example in *numbers,*—the *fourth root* of the square of 9, that is, of 81, is 3, and the square root of 9 is also 3.

On this principle a fractional exponent may always be taken in its *lowest terms.*

The Sign to be Prefixed to a Root.

(214.) Every *odd* Root of a quantity has the same sign as the *quantity itself.*

For the quantity itself is an *odd power* with reference to such *root,* (206), and an *odd* power has the same sign as the quantity of which it is a power, (202).

(215.) Every *even* Root of a *positive* quantity is *ambiguous,* that is, the quantity itself does not determine whether the root is *positive* or *negative.*

This follows from the principle that every *even power* of a *negative,* as well as of a positive quantity, is *positive,* (202).

The square of $-a$, as well as of $+a$, is $+a^2$; hence the square *root* of a^2 is $+a$ or else $-a$; and when it is not known whether a^2 was derived from a or $-a$, it is uncertain which of these two is the square root.

This uncertainty as to the *sign* of the root, is expressed by the *ambiguous* sign ± *plus* or *minus;* thus

the square root of a^2 is $\pm a$.

For a like reason the *fourth* root of a^4 is $\pm a$.

The square root of 9 is ± 3; the fourth root of 16 is ± 2; &c.

(216.) An *even* Root of a *negative* quantity is *impossible,* since there is no quantity which can be raised to an *even negative power,* (202).

Thus no quantity multiplied into itself will produce $-a^2$; this quantity therefore has *no square root.* For a like reason $-a^4$ has no square root or *fourth root.*

In like manner -9 has no square root; -16 has no square root or fourth root; $-25a^2$ has no square root, &c.

EXERCISES

On the Roots of Monomials.

1. Find the square root of $4a^2x^4$. *Ans.* $\pm 2ax^2$.
2. Find the cube root of $8a^3y^6$. *Ans.* $2ay^2$.
3. Find the square root of $9a^4x$. *Ans.* $\pm 3a^2x^{\frac{1}{2}}$.
4. Find the cube root of $-27y^2$. *Ans.* $-3y^{\frac{2}{3}}$.
5. Find the square root of $16x^{\frac{1}{2}}$. *Ans.* $\pm 4x^{\frac{1}{4}}$.
6. Find the cube root of a^3xy^2. *Ans.* $ax^{\frac{1}{3}}y^{\frac{2}{3}}$.
7. Find the square root of $25a^3$. *Ans.* $\pm 5a^{\frac{3}{2}}$.
8. Find the cube root of $-64y^2$. *Ans.* $-4y^{\frac{2}{3}}$
9. Find the square root of $36x^6$. *Ans.* $\pm 6x^3$.
10. Find the cube root of $8ay^n$. *Ans.* $2a^{\frac{1}{3}}y^{\frac{n}{3}}$.
11 Find the square root of $\frac{1}{4}a^4y^2$. *Ans.* $\pm\frac{1}{2}a^2y$.
12 Find the cube root of $-8a^3x^{3n}$. *Ans.* $-2ax^n$.
13. Find the square root of $\frac{4}{9}xy^4$. *Ans.* $\pm\frac{2}{3}x^{\frac{1}{2}}y^2$.
14. Find the cube root of $-\frac{1}{8}ax^3$. *Ans.* $-\frac{1}{2}a^{\frac{1}{3}}x$.
15. Find the fourth root of $16a^4y^2$. *Ans.* $\pm 2ay^{\frac{1}{2}}$.
16. Find the cube root of $-\frac{8}{27}x^2$. *Ans.* $-\frac{2}{3}x^{\frac{2}{3}}$.
17. Find the fourth root of $\frac{1}{16}ay^2$. *Ans.* $\pm\frac{1}{2}a^{\frac{1}{4}}y^{\frac{1}{2}}$.
18. Find the fifth root of $-a^{10}x^2y$. *Ans.* $-a^2x^{\frac{2}{5}}y^{\frac{1}{5}}$.
19. Find the sixth root of a^3x^2y. *Ans.* $\pm a^{\frac{1}{2}}x^{\frac{1}{3}}y^{\frac{1}{6}}$.
20. Find the sixth root of ba^2y^{3n}. *Ans.* $\pm b^{\frac{1}{6}}a^{\frac{1}{3}}y^{\frac{n}{2}}$.

Roots of Polynomials.

(217.) The method of extracting any required Root of a Polynomial, may be discovered from the manner in which the corresponding *power* of a polynomial is formed. This subject will be elucidated under the appropriate Rules.

RULE XVIII.

(218.) *To Extract the Square Root of a Polynomial.*

1. Arrange the Polynomial according to the powers of one of its letters, and take the square root of the *left hand term*, for the first term of the root.

2. Subtract the square of the root thus found from the given Polynomial; divide the remainder by *twice the root* already found, and annex the *quotient* to both the *root* and the *divisor*.

3. Multiply the divisor thus formed by the *last term in the root;* subtract the product from the dividend; divide the remainder by *twice the root* now found; and so on, as before.

EXAMPLE.

To extract the Square Root of

$$a^2+2ab+b^2.$$

The required Root, we already know, is $a+b$, since the *square* of this binomial is the given trinomial:—our object is to show that the root $a+b$ would be found by the foregoing Rule.

$$\begin{array}{rl} & a^2+2ab+b^2(a+b \\ & a^2 \\ \cline{2-2} 2a+b) & \quad 2ab+b^2 \\ & \quad 2ab+b^2 \\ \cline{2-2} \end{array}$$

The first term a of the root, is the square root of a^2, the left hand term of the given polynomial. Subtracting the square of a, we have the remainder $2ab+b^2$.

We have now to discover a *divisor* of this remainder, which will give, for a quotient, the next term b of the root.

$2a$, that is, *twice the root* already found, divided into $2ab$, gives b; and b annexed to $2a$ makes the divisor $2a+b$. This divisor multiplied by b, produces $2ab+b^2$,—which completes the operation.

The Rule is framed in accordance with the process thus discovered.

EXERCISES

On the Square Root of Polynomials.

1. Find the square root of the polynomial
$a^4-4a^3+6a^2-4a+1$. *Ans.* a^2-2a+1.

2. Find the square root of the polynomial
$a^4+4a^3x+6a^2x^2+4ax^3+x^4$. *Ans.* $a^2+2ax+x^2$.

3. Find the square root of the polynomial
$1-6y+13y^2-12y^3+4y^4$. *Ans.* $1-3y+2y^2$.

4. Find the square root of the polynomial
$4x^6-4x^4+12x^3+x^2-6x+9$. *Ans.* $2x^3-x+3$.

5. Find the square root of the polynomial
$4a^4+12a^3x+13a^2x^2+6ax^3+x^4$. *Ans.* $2a^2+3ax+x^2$.

6. Find the square root of the polynomial
$1-6a+13a^2-12a^3+4a^4$. *Ans.* $1-3a+2a^2$.

7. Find the square root of the polynomial
$x^4+2ax^3+3a^2x^2+2a^3x+a^4$. *Ans.* x^2+ax+a^2.

8. Find the square root of the polynomial
$9a^4-12a^3y+28a^2y^2-16ay^3+16y^4$. *Ans.* $3a^2-2ay+4y^2$.

9. Find the square root of the polynomial
$25+34x^2+12x^3+20x+9x^4$. *Ans.* $5+2x+3x^2$.

10. Find the square root of the polynomial
$a^4+4a^3y-12ay+4a^2y^2+9-6a^2$. *Ans.* $a^2+2ay-3$.

11. Find the square root of the polynomial
$4+24x^2-16x+4x^4-16x^3$. *Ans.* $2-4x+2x^2$.

12. Find the square root of the polynomial
$1-2y+7y^2-2y^3+5y^4+12y^5+4y^6$. *Ans.* $1-y+3y^2+2y^3$.

Square Root of Numbers.

The preceding Rule is substantially the same as the Arithmetical Rule for extracting the square root of Numbers. To show its application, however, to the latter purpose, we must premise the following principles.

(219.) If a Number be separated into *periods of two figures* each,—from right to left,—these periods will correspond, respectively, to the *units*, *tens*, *hundreds*, &c., in the Square Root of the number.

For since the square of *ten* is 100, the square of the *tens figure* in the root will leave *two vacant* places in the right of the given number; these two places must therefore correspond to the *units* in the root.

And since the square of *a hundred* is 10000, the square of the *hundreds* in the root leaves *four vacant* places in the right of the number; and the first two corresponding to the *units*, the *next two* must correspond to the *tens* in the root.

In like manner it is shown that the *third period* of two figures corresponds to the *hundreds* in the square root of the given number; and so on.

(220.) If a Number be divided into any two parts, the Square of the number will be equal to the *square of the first part* + *twice the first* × *the second* + the *square of the second part*.

For, a representing the *first part* of the number, and b the *second*, $a+b$ will be equal to the number; and

$$(a+b)^2=a^2+2ab+b^2.$$

With these principles established we may proceed to the following

EXAMPLES.

1. To extract the Square Root of 529

$$
\begin{array}{rl}
 & 5'29\;(\;23 \\
 & \underline{4} \\
43) & 129 \\
 & \underline{129}
\end{array}
$$

The first period 29 corresponds to the *units* figure, and the 5 to the *tens* figure in the root, (219).

The greatest *integral square root* of the 5 is 2 *tens*, the square of which is 4 *hundreds*, and this subtracted leaves 129.

Since the root of the given number consists of two parts, *tens* and *units*, and the *square of the* 2 *tens* has been subtracted, the remainder

129 = *twice* 2 *tens* × the *units* + the *units*^{2}, (220).

Doubling the root, 2 tens, already found, we take the product, 4 *tens*, for a divisor. The 0 in 4 tens, or 40, may be omitted in finding the next figure in the root, if we omit the 9 in the corresponding place in the 129.

We therefore say 4 in 12, 3 *times*. This 3 annexed to the 4 makes the 4 now become 4 tens. Then 43, or 4 tens + 3 units, × 3 *units*, produces the remainder 129; and thus completes the operation.

2. To extract the Square Root of 36 84 49

36'84'49 (607
36
1207) 84 49
84 49

Pointing off periods of two figures, from right to left, we find that the root will contain *units*, *tens*, and *hundreds*, (219).

The square root of the left hand period 36, is 6, the square of which subtracted leaves no remainder in that period.

We take the next period 84 to find the next figure in the root. Doubling the root 6, for a divisor, and omitting the 4, we say 12 in 8, 0 time.

Including the next period 49, and doubling the root 60 now found, for a divisor, we say 120 in 844, 7 times.

Annexing 7 to 120, the divisor becomes 1207, which multiplied by 7 produces 8449; and thus the operation is completed.

Square Root of Decimals.

(221.) In extracting the Square Root of a Decimal Fraction, the *periods* must be taken from the decimal point *towards the right;* and a 0 must be annexed, if necessary, to complete the last period.

The last period must be complete, because, by the principles of decimal multiplication, the square of a decimal Fraction must contain *twice as many* decimal figures as are in the root.

The number of decimal figures to be made in the root, is therefore the same as the number of *decimal periods.*

When an exact root cannot be found, decimal periods of 00 each may be annexed, and the root continued in *decimals* to any required exactness.

EXERCISES

On the Square Root of Numbers.

1. Find the square root of 11236, and of $\frac{256}{289}$.
Ans. 106; and $\frac{16}{17}$.

2. Find the square root of 38809, and of $\frac{1156}{5041}$.
Ans. 197; and $\frac{34}{71}$.

3. Find the square root of 75076, and of $\frac{6561}{9025}$.
Ans. 274; and $\frac{81}{95}$.

4. Find the square root of 22801, and of $\frac{5929}{7056}$.
Ans. 151; and $\frac{77}{84}$.

5. Find the square root of .582 169, and of $127\frac{13}{25}$.

Reduce the $\frac{13}{25}$ to an eqivalent *decimal,* before proceeding with the evolution.

Ans. .763; and 11.292'.

6. Find the square root of 475.125, and of $346\frac{17}{30}$.
Ans. 21.797'; and 18.616'.

7. Find the square root of 37.4780, and of $470\frac{21}{100}$.
Ans. 6.121'; and 21.684'

RULE XIX.

(222.) *To Extract the cube Root of a Polynomial.*

1. Arrange the Polynomial according to the powers of one of its letters, and take the cube root of the *left hand term,* for the first term of the root.

2. Subtract the cube of the root thus found from the given Polynomial, and divide the remainder by 3 *times the square* of the root already found, and annex the *quotient* to the root.

3. *Complete the divisor* by adding to it the product which arises from annexing the last term in the root to 3 times the *other part* of the root, and *multiplying the result* by the last term.

4. Multiply the completed divisor by the *last term* in the root, and subtract the product from the dividend.

5. Find the next *incomplete divisor* by adding to the last complete divisor the *product which completed it,* and the *square of the last term* in the root; divide, and complete the divisor, as before; and so on.

EXAMPLE.

To extract the Cube Root of

$$a^3+3a^2b+3ab^2+b^3.$$

The required Root, we already know, is $a+b$, (203), and our object is to show that this root would be found by the Rule.

$$\begin{array}{lll} & a^3+3a^2b+3ab^2+b^3 & (\ a+b \\ & a^3 & \\ \cline{2-2} 3a^2+3ab+b^2 &)\quad 3a^2b+3ab^2+b^3 & \\ & \quad\ \underline{3a^2b+3ab^2+b^3} & \end{array}$$

The first term a of the root, is the cube root of a^3, the left hand term of the given polynomial. Subtracting a^3, we have the remainder $3a^2b+3ab^2+b^3$.

We have now to find a *divisor* of this remainder, which will give, for a quotient, the next term b of the root.

$3a^2$, that is, 3 *times the square* of the root already found, divided into $3a^2b$, gives b. Adding $3ab$, and also b^2, to $3a^2$, the completed divisor is $3a^2+3ab+b^2$.

This divisor multiplied by b, produces the last dividend, and thus shows that the operation is completed.

EXERCISES

On the Cube Root of Polynomials.

1. Find the cube root of the polynomial
$a^6-6a^5+15a^4-20a^3+15a^2-6a+1$. *Ans.* a^2-2a+1.
2. Find the cube root of the polynomial
$1-6x+21x^2-44x^3+63x^4-54x^5+27x^6$. *Ans.* $1-2x+3x^2$.
3. Find the cube root of the polynomial
$8-12x+30x^2-25x^3+30x^4-12x^5+8x^6$. *Ans.* $2-x+2x^2$.
4. Find the cube root of the polynomial
$8+36y^2+24y+32y^3+6y^5+y^6+18y^4$. *Ans.* $2+2y+y^2$,
5. Find the cube root of the polynomial
$9a^4-3a^5-13a^3+a^6-12a+8+18a^2$. *Ans.* a^2-a+2.
6. Find the cube root of the polynomial
$21x^4-44x^3-54x+63x^2-6x^5+x^6+27$. *Ans.* x^2-2x+3.
7. Find the cube root of the polynomial
$y^6-9y^5+39y^4-99y^3+156y^2-144y+64$. *Ans.* y^2-3y+4

CUBE ROOT OF NUMBERS.

The method of extracting the Cube Root of Numbers involves the following principles.

(223.) If a Number be separated into *periods of three figures* each, —from right to left,—these periods will correspond, respectively, to the *units, tens, hundreds,* &c., in the Cube Root of the number.

For since the cube of *ten* is 1000, the cube of the *tens figure* in the root, will leave three vacant places in the right of the given number; these three places must therefore correspond to the *units* in the root.

And since the cube of *a hundred* is 1000000, the cube of the *hundreds* in the root leaves *six vacant* places in the right of the number; and the first three corresponding to the *units*, the *next three* must correspond to the *tens* in the root.

In like manner it is shown that the *third period* of three figures corresponds to the *hundreds* in the cube root of the given number, and so on.

(224.) If a Number be divided into any two parts, the Cube of the number will be equal to the *cube of the first part* + 3 *times the square of the first* × the *second* + 3 *times the first* × the *square of the second* + the *cube of the second.*

For a representing the *first part* of the number, and b the *second*, $a+b$ will be equal to the number; and

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3.$$

As an example of the application of this principle to numbers, let 275 be divided into the two parts 200 and 75; then

$$(200+75)^3=200^3+3\times200^2\times75+3\times200\times75^2+75^3.$$

If the values of the several terms on the right be computed, and added together, we shall obtain the cube of 275.

We might now proceed to extract the Cube Root of Numbers in a manner corresponding to the Rule already given for the cube root of Polynomials; but the method by the following Rule is more direct and simple.

RULE XX.

(225.) *To Extract the Cube Root of a Number.*

1. Separate the number into *periods of three figures* each, from right to left, observing that the last period will sometimes have but one or two figures.

2. Take the greatest integral cube root of the left hand period, for the first figure of the root required,—subtract *its cube* from said period,—and to the remainder affix the next period for a *dividend.*

3. Take 3 *times the square* of the root already found, for an incomplete divisor; divide it into the dividend *exclusive of its two right hand figures,* and annex the quotient to the root.

4. *Complete the divisor* by annexing to it 00, and adding the product which arises from annexing the last figure in the root to 3 *times the other part* of the root, and multiplying the *result by the last figure.*

5. Multiply the completed divisor by the last figure in the root; subtract the product from the dividend; and to the remainder affix the next period for a new dividend.

6. Find the *next incomplete divisor* by adding to the last complete divisor the *product which completed it,* and the *square of the last figure* in the root; divide, and complete the divisor, as before; and so on.

In applying this Rule it will be convenient to have the following

Table of Roots and Cubes.

Roots,	1, ...	2, ...	3, ...	4, ...	5, ...	6, ...	7, ...	8, ...	9.
Cubes,	1, ...	8, ...	27, ...	64, ..	125, ..	216, ..	343, ..	512, ..	729.

We will first apply, and then demonstrate this Rule.

EXAMPLE.

To extract the Cube Root of 95 44 39 93.

	95'44 3'9 93 (467
	64
48	31 44 3
4800+625=54 25	27 12 5
5425+ 625+ 25=60 75	43 1 8 993
607500+ 9499= 6169 99	43 1 8 993

The greatest integral cube root of the left hand period 95, is 4, the *cube* of which subtracted leaves the remainder 31. Affixing the next period, we obtain

the dividend 31443.

Three times the square of 4, the root already found, is 48. We divide 48 into 314, excluding the 43. The quotient would appear to be 6; but the divisor being as yet *too small*, we take 5 for the quotient.

To *complete the divisor*, we annex 00 to it, and add 625;—the 625 being obtained by annexing the 5 in the root to 3 times the 4, and multiplying the result 125 by 5.

Multiplying the completed divisor 5425 by 5, subtracting, and affixing the next period to the remainder, we find

the dividend 43 18 99 3.

For the *next incomplete divisor*, we add to the last complete divisor the product 625 which completed it, and the square 25 of the 5 in the root. Dividing by 6075, the quotient is 7.

To *complete this divisor*, we annex 00, and add 9499,—this last number being obtained by annexing the 7 to 3 times the 45, and multiplying the result 1357 by 7. The divisor now found multiplied by 7 produces the last dividend.

Demonstration of the Rule for the Cube Root of Numbers.

To facilitate the Demonstration, we will take a number containing but *two periods*,—whose root will therefore consist of *tens* and *units*, (223).

	91'125 (45
	64
48	27125
4800+625 =5425	27125

The 4 in the root, found according to the Rule, we know is 4 *tens*, (223); and the second figure in the root will be *units*.

Let t represent the *tens*, and u the *units* of the root, then

the *given number* $=t^3+3t^2u+3tu^2+u^3$, (224).

Hence the cube 64,—which is 64 *thousand*,—of the 4 *tens*, subtracted, leaves $27125=3t^2u+3tu^2+u^3$.

By the Rule for the cube root of a Polynomial, the *incomplete divisor* of this remainder—to be used for finding the next figure or term u of the root, is $3t^2=3$ times (4 *tens*)$^2=4800$.

In dividing we omit the 00, and at the same time exclude the 25 in the corresponding places of the dividend 27125.

By the Rule just referred to, the quantity to be added to complete the divisor $3t^2$, is

$$(3t+u)u=(3\times 4\ tens\ +5)\times 5=625;$$

that is, the product which arises from annexing the last figure in the root to 3 *times the other part* of the root, and *multiplying the result* by the last figure.

Suppose now that the given number contains *three periods*, and its root, consequently, three figures.

Regarding the two figures already found as constituting the *first part* of the root, the next *incomplete* divisor, by Rule XIX, would be

$$3t^2+(3t+u)u+(3t+u)u+u^2;$$

which consists, first, of the last complete divisor $3t^2+(3t+u)u$; secondly, of the product $(3t+u)u$ which was added to $3t^2$ to complete that divisor; thirdly, of the square of the last figure u in the root.

In like manner the Rule may be shown to be applicable, whatever be the number of figures in the required root.

Cube Root of Decimals.

(226.) In extracting the Cube Root of a Decimal Fraction, the *periods* must be taken from the decimal point *towards the right*, and 0 or 00 must be annexed, if necessary, to complete the last period.

The last period must be complete, because, by the principles of decimal multiplication, the cube of a decimal Fraction must contain 3 *times as many* decimal figures as are in the root.

The number of decimal figures to be made in the root, is therefore the same as the number of *decimal periods.*

When an exact root cannot be found, decimal periods of 000 each may be annexed, and the root *continued in decimals* to any required exactness.

EXERCISES

On the Cube Root of Numbers.

1. Find the cube root of 262144, and of $\frac{19683}{389017}$.
Ans. 64; and $\frac{27}{73}$.

2. Find the cube root of 2406104, and of $\frac{24389}{438976}$.
Ans. 134; and $\frac{29}{76}$.

3. Find the cube root of 22906304, and of $\frac{29791}{681472}$.
Ans. 284; and $\frac{31}{88}$

4. Find the cube root of 479.2735, and of $8377\frac{23}{150}$.
Ans. 7.825'; and 20.309'.

5. Find the cube root of 5371.3745, and of $3059\frac{31}{200}$.
Ans. 17.513'; and 14.51'

6. Find the cube root of 403.73331, and of $.71200\frac{40}{123}$.
Ans. 7.390'; and .892'.

7. Find the cube root of 4370.666, and of $\frac{4913}{5832}$.
Ans. 16.34'; and $\frac{17}{18}$.

8. Find the cube root of 20796875, and of $3511\frac{101}{125}$.
Ans. 275; and 15.20.

9. Find the cube root of .202262003, and of $\frac{2744}{6859}$.
Ans. .587; and $\frac{14}{19}$.

10. Find the cube root of 103823, and of $2460\frac{3}{8}$.
Ans. 47; and 13.5.

EXTRACTION OF THE *n*th ROOT.

(227.) Any Root whatever of a Polynomial might be extracted,—by taking the root of its left hand term,—with this root forming an *incomplete* divisor,—with the quotient term, and the root already found, *completing* the divisor,—and so on, in a manner depending on the *order of the root* to be extracted.

But this method, which is preferable for the *square* and *cube*, becomes too complicated when applied to the higher roots.

By dispensing with the completed divisors, the operation may be simplified, and the process of Evolution *generalized*, as follows.

RULE XXI.

(228.) *To Extract any Root of a Polynomial.*

1. Uunderstanding the *order of the root* to be denoted by n,—arrange the Polynomial according to the powers of one of its letters, and take the nth root of its left hand term, for the first term of the root.

2. Subtract the nth *power* of the root found from the given Polynomial; and divide the remainder by n times the $(n-1)$ power of this root, for the second term of the root.

3. Subtract the nth power of the root now found from the given Polynomial, and, using the same divisor as before, proceed in the same manner till the nth power of the root becomes equal to the given Polynomial.

This Rule may also be applied to Numbers, by taking n figures in each period, from right to left, for *integers*, and from the decimal point towards the right, for *decimals;* and there will be less liability to error in finding the *quotient figure*, if new *divisors* be found for the second and subsequent remainders.

EXAMPLE.

To extract the 4th root of 30 49 800 625.

$$\begin{array}{rll} & 30'4980'0625\ (\ 235 & \\ 2^4 = & 16 & \\ \hline 4\times 2^3 = 32\ ) & 144980 & \ldots\ \text{Exclude 980 in dividing.} \\ \hline 23^4 = & 279841 & \\ \hline 4\times 23^3 = 48668\ ) & 251390625 & \ldots\ \text{Exclude 625 in dividing.} \\ \hline 235^4 = & 3049800625 & \end{array}$$

The preceding root might also be found by two extractions of the *square root*, (210); thus

The square root of 3049800625 is 55225; and the square root of the latter number is 235.

CHAPTER IX.

IRRATIONAL OR SURD QUANTITIES.—IMAGINARY QUANTITIES

Perfect and Imperfect Powers.

(229.) A Perfect Power, of any degree, is a quantity which has an exact *root* of the same degree;—otherwise, the quantity is called an *Imperfect Power*.

Thus 4 and $9a^2$ are perfect *squares*, having the exact square *roots* 2 and $3a$; while $2a$ and $8x^3$ are *imperfect* squares, since they have no exact square roots.

In like manner 8 and $27a^3$ are perfect *cubes;* while 9 and $25x^2$ are imperfect cubes, since they have no exact cube roots.

The Polynomial $a^2+2ab+b^2$ is a perfect square, whose root is $a+b$; (218); and $a^3-3a^2x+3ax^2-x^3$ is a perfect cube, whose root is $a-x$.

IRRATIONAL OR SURD QUANTITIES.

(230.) A *Rational* quantity is one which can be accurately expressed without any *indicated root;* as 2, $3a$, or $\frac{2}{3}x$.

An *Irrational* or *Surd* quantity is one which can be accurately expressed only under the form of a *root*,—being the indicated *root* of an *imperfect power*.

Thus $2^{\frac{1}{2}}$ is an *irrational* quantity, since the exact square root of 2 cannot be determined. Also $a^{\frac{1}{2}}$ and $x^{\frac{1}{3}}$ are irrational quantities.

By the common Rule for the square root of numbers, we should find the square root of $2=1.414213'$; but other decimal figures would succeed without end.

The term *irrational* when applied to a quantity, implies that such quantity has no determinable *ratio* to unity.

Irrational or Surd quantities—being expressed under the form of *roots*—are also called *Radical* quantities, from the Latin *Radix*, a *root*

Radical Sign.

(231.) The *radical sign* $\sqrt{\ }$ prefixed to a quantity, denotes the *square root* of the quantity; and is therefore equivalent to the *exponent* $\frac{1}{2}$.

Thus $\sqrt{a}$ is the square root of a; equivalent to $a^{\frac{1}{2}}$.

With the *index* 3 affixed to it, this sign denotes the *cube root*; and is then equivalent to the *exponent* $\frac{1}{3}$. With the index 4, it denotes the 4th root; and so on.

Thus $\sqrt[3]{a}$ is equivalent to $a^{\frac{1}{3}}$; and $\sqrt[4]{a}$ to $a^{\frac{1}{4}}$.

The radical sign may always be thus superseded by a *fractional exponent;* and this should be done whenever any obscurity would arise in calculation from using this sign.

For example, in extracting the square root of $\sqrt[3]{a}$, we substitute the exponent $\frac{1}{3}$, and find the required root to be

$$a^{\frac{1}{3}\div 2}=a^{\frac{1}{6}};\ (208).$$

A quantity preceding the $\sqrt{\ }$, without $+$ or $-$ interposed, is a *coefficient* or multiplier of the Surd; and when no coefficient is expressed, a *unit* is understood.

Thus $5\sqrt{a}$ is 5 *times* the square root of a; and $\sqrt{ax}$ is the same as $1\sqrt{ax}$.

Similar and Dissimilar Surds.

(232.) *Similar* Surds are such as express the *same root* of the same quantity; otherwise, the Surds are *dissimilar.*

Thus $\sqrt{3}$ and $5\sqrt{3}$ are *similar;* while $\sqrt[3]{3}$ and $5\sqrt{3}$ are dissimilar. So $(a+b)^{\frac{1}{2}}$ and $3\sqrt{a+b}$ are similar Surds.

A Rational Quantity under the Form of a Surd.

(233.) A rational quantity may be expressed under the form of the *square root*, or cube root, &c., by placing the corresponding *power* of the quantity under the *exponent* or *sign* of the root.

To express $3a$ under the form of the *square root*, we place the square of $3a$, which is $9a^2$, under the *exponent* or *sign* of the square root, thus

$$3a=(9a^2)^{\frac{1}{2}} \text{ or } \sqrt{9a^2}.$$

TRANSFORMATION OR REDUCTION OF SURDS.

Certain Transformations of radical quantities are sometimes necessary, to adapt them to the purposes of calculation. These depend chiefly on the two following principles.

(234.) When two or more Factors have the same *exponent*, that exponent may be transferred to the *product* of those factors; and,

Conversely, the exponent of a Product may be transferred to *each of the factors* which compose that product.

Thus a^2x^2 is equal to $(ax)^2$; the product of the *squares* of a and x, is equal to the square of the *product* of a and x.

And $a^{\frac{1}{2}}x^{\frac{1}{2}}$ is equal to $(ax)^{\frac{1}{2}}$; the product of the *square roots* of a and x, is equal to the square root of the *product* of a and x.

These principles result from the methods of finding the *powers* and *roots* of quantities; thus by squaring ax,—and extracting the square root of ax,—we have

$$(ax)^2=a^2x^2,\ (200);\ \text{and}\ (ax)^{\frac{1}{2}}=a^{\frac{1}{2}}x^{\frac{1}{2}},\ \ (208).$$

These methods of illustration may be applied to any other *power* or *root* as well as to the square and square root. Thus the product of the *cube roots* of two or more factors is equal to the cube root of the *product* of those factors; and so for any power or root.

To give examples in numbers;—

$$\sqrt{4}\times\sqrt{25}=\sqrt{100};\ \text{or}\ 2\times5=10.$$
$$\sqrt[3]{8}\times\sqrt[3]{27}=\sqrt[3]{216};\ \text{or}\ 2\times3=6.$$

(235.) The Exponent of a quantity may be changed to any *equivalent expression*, without altering the value of the power or root denoted.

This proposition is substantially the same as one already demonstrated, (213).

As examples, $a^{\frac{1}{2}}=a^{\frac{2}{4}}=a^{\frac{3}{6}}=a^{\frac{4}{8}}$;

$$4^{\frac{1}{2}}=4^{\frac{2}{4}}=16^{\frac{1}{4}}=2.$$

The following are the principal Transformations required in the calculation of Irrational or Surd quantities.

Product of a Rational and an Irrational Factor.

(236.) A *rational* multiplier or *coefficient* may be put under the form of the Surd annexed, and the product of the two radicals be then affected with the same fractional exponent or radical sign.

Thus $a\sqrt{3}=\sqrt{a^2}\sqrt{3}=\sqrt{3a^2}$, by putting the coefficient a under the form of the *square root*, (233), and transferring the sign $\sqrt{}$, which is equivalent to the *exponent* $\frac{1}{2}$, from the two factors to the product $3a^2$, (234).

In like manner $2\sqrt{5}=\sqrt{4}\times\sqrt{5}=\sqrt{20}$.

As an example of the *utility* of this Transformation,—suppose it were required to find an *approximate value* of $2\sqrt{5}$.

By the common method of extracting the square root, we might find an approximate value of $\sqrt{5}$, and such value multiplied by 2 would be an approximate value of $2\sqrt{5}$.

It is evident, however, that this process would *double*, in the product, the *deficiency* in the value found of $\sqrt{5}$; and the error thus arising will be greater as the coefficient of the Surd is greater.

By taking $\sqrt{20}$, instead of $2\sqrt{5}$, the approximate value will be found independently of the preceding *source of error*.

Surds Reduced to Simpler Forms.

(237.) A Surd is *simplified* by resolving the quantity under the $\sqrt{}$, or exponent, into two factors—one of which is a *perfect power* of like degree—and putting the extracted *root* of this factor as a coefficient to the indicated root of the other.

$$\text{Thus } \sqrt{50}=\sqrt{25}\times\sqrt{2}=5\sqrt{2};$$

$$\text{or, } 50^{\frac{1}{2}}=25^{\frac{1}{2}}\times 2^{\frac{1}{2}}=5\times 2^{\frac{1}{2}}, \text{ (234).}$$

In like manner, $\sqrt{4a^2+8a^3}=\sqrt{4a^2}\sqrt{1+2a}=2a\sqrt{1+2a}$.

In these examples the given Surds are *square roots*, and the factors 25 and $4a^2$ are accordingly perfect *squares*.

As an example of the *cube root*, we have

$$\sqrt[3]{54}=\sqrt[3]{27}\times\sqrt[3]{2}=3\sqrt[3]{2};\ \text{27 being a perfect } \textit{cube}.$$

When the given Surd is preceded by a Coefficient, this coefficient must be multiplied into the one found by the process of simplifying.

$$\text{Thus } 3\sqrt{32a^3}=3\sqrt{16a^2}\sqrt{2a}=3\times 4a\sqrt{2a}=12a\sqrt{2a}.$$

If, in the preceding process, the square or cube factor be the *greatest* square or cube thus entering into the composition of the quantity under the $\sqrt{}$, the Surd will be reduced to its *simplest form.*

Thus in each of the examples above given, the Surd is reduced to its simplest form. The second example would also give

$$\sqrt{4a^2+8a^3}=\sqrt{4}\sqrt{a^2+2a^3}=2\sqrt{a^2+2a^3}\,;$$

which is not the simplest form, since a^2 is still under the $\sqrt{}$.

(238.) A Fractional Surd may be reduced to an *integral surd*, by multiplying both its terms, if necessary, to make the denominator a *perfect power*, and then resolving into factors, and proceeding as before, (237).

$$\text{For example, } \sqrt{\frac{3}{8}}=\sqrt{\frac{6}{16}}=\sqrt{\frac{1}{16}}\,\sqrt{6}=\frac{1}{4}\,\sqrt{6}\,;$$

We multiply both terms of the first Fraction by 2, to make the denominator 16 a perfect *square.* The reciprocal of this denominator is the square factor, and the numerator 6 is the other factor.

Surds which are apparently *dissimilar*, often become *similar* when reduced to their simplest forms. They are thus prepared for Addition or Subtraction, as will be seen hereafter.

Surds of Different Roots reduced to the Same Root.

(239.) Two or more Surds of *different roots*, may be reduced to equivalent ones of the *same root*,—by reducing their fractional exponents to a *common denominator*,—raising each quantity under the $\sqrt{}$ to the power denoted by the *numerator* of its new exponent,—and taking the root denoted by the common denominator.

Thus to reduce $\sqrt{5}$ and $\sqrt[3]{4}$ to the *same root.*

Reducing the exponents $\frac{1}{2}$ and $\frac{1}{3}$ to a common denominator,

we find $\sqrt{5}=5^{\frac{3}{6}}$, and $\sqrt[3]{4}=4^{\frac{2}{6}}$, (235).

Cubing the 5, and *squaring* the 4, according to the numerators of their new exponents, (211), we find

$$\sqrt{5}=125^{\frac{1}{6}},\text{ and }\sqrt[3]{4}=16^{\frac{1}{6}}.$$

The *square* root and the *cube* root have thus been both reduced to the *sixth* root.—This kind of reduction is sometimes necessary in the Multiplication and Division of Surds,—as will be seen hereafter.

EXERCISES

On the Transformation or Reduction of Surds.

1. Find the Product of $3\sqrt{4a}$. (236). *Ans.* $\sqrt{36a}$
2. Find the Product of $2\sqrt[3]{3x}$. *Ans.* $\sqrt[3]{24x}$.
3 Find the Product of $4\sqrt{ab}$. *Ans.* $\sqrt{16ab}$.
4. Find the Product of $a\sqrt[3]{5x}$. *Ans.* $\sqrt[3]{5a^3x}$.
5. Find the Product of $3x\sqrt{7}$. *Ans.* $\sqrt{63x^2}$
6. Find the Product of $ax\sqrt[3]{10}$. *Ans.* $\sqrt[3]{10a^3x^3}$.
7. Find the Product of $7\sqrt{a+1}$. *Ans.* $\sqrt{49a+49}$.
8 Find the Product of $2\sqrt[3]{1-x}$. *Ans.* $\sqrt[3]{8-8x}$.
9 Find the Product of $(a+b)\sqrt{2}$. *Ans.* $\sqrt{2(a+b)^2}$.
10. Find the Product of $(a-x)\sqrt[3]{3}$. *Ans.* $\sqrt[3]{3(a-x)^3}$.
11. Find the Product of $(a+1)\sqrt{x}$. *Ans.* $\sqrt{x(a+1)^2}$.
12. Find the Product of $(a-1)\sqrt[3]{y}$. *Ans.* $\sqrt[3]{y(a-1)^3}$

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13. Reduce $\sqrt{245ay^5}$ to its *simplest form.* (237).

To discover the greatest *square factor* of 245, we will divide this number successively by the *square numbers*,

4, 9, 16, 25, 36, 49, 64, &c.,

and take the largest divisor that leaves no *remainder.*

Such divisor will be found to be 49;—49)245(5.

Then $\sqrt{245ay^5}=\sqrt{49y^4}\sqrt{5ay}=7y^2\sqrt{5ay}$.

14. Reduce $\sqrt{4a^2b}$ to its simplest form. *Ans.* $2a\sqrt{b}$.
15. Reduce $3\sqrt{a^3x}$ to its simplest form. *Ans.* $3a\sqrt{ax}$
16. Reduce $\sqrt[3]{8y}$ to its simplest form. *Ans.* $2\sqrt[3]{y}$
17. Reduce $2\sqrt{8ax^2}$ to its simplest form. *Ans* $4x\sqrt{2a}$
18. Reduce $\sqrt[3]{27a^4}$ to its simplest form. *Ans.* $3a\sqrt[3]{a}$.
19 Reduce $5\sqrt{48a^3y}$ to its simplest form. *Ans.* $20a\sqrt{3ay}$.
20. Reduce $2\sqrt[3]{64a^4}$ to its simplest form. *Ans.* $8a\sqrt[3]{a}$.

21. Reduce $3\sqrt{128x^3}$ to its simplest form. *Ans.* $24x\sqrt{2x}$

22. Reduce $a\sqrt[3]{250y}$ to its simplest form. *Ans.* $5a\sqrt[3]{2y}$

23. Reduce $2x\sqrt{432}$ to its simplest form. *Ans.* $24x\sqrt{3}$.

24. Reduce $3y\sqrt[3]{135}$ to its simplest form. *Ans.* $9y\sqrt[3]{5}$.

25. Reduce $\sqrt{8+12a^3}$ to its simplest form. *Ans.* $2\sqrt{2+3a^3}$.

26. Reduce $(a+b)\sqrt[3]{81x^3}$ to its simplest form. *Ans.* $3x(a+b)\sqrt[3]{3}$

27. Reduce $3\sqrt{a^2x-2a^3}$ to its simplest form. *Ans.* $3a\sqrt{x-2a}$.

28. Reduce $(a-x)\sqrt[3]{192a^3x}$ to its simplest form. *Ans.* $4a(a-x)\sqrt[3]{3x}$.

29. Reduce $4\sqrt{4x^2+8x^3}$ to its simplest form. *Ans.* $8x\sqrt{1+2x}$.

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30 Reduce $3\sqrt{\frac{27}{50}}$ to an *integral* Surd in its *simplest form.* (238).

$$3\sqrt{\frac{27}{50}}=3\sqrt{\frac{27\times2}{50\times2}}=3\sqrt{\frac{54}{100}}=3\sqrt{\frac{9}{100}}\sqrt{6}=\frac{9}{10}\sqrt{6}.$$

31 Reduce $2\sqrt{\frac{14}{25}}$ to an integral Surd in its simplest form.

Ans. $\frac{2}{5}\sqrt{14}$

32 Reduce $4\sqrt{\frac{4ax^3}{11}}$ to an integral Surd in its simplest form.

Ans. $\frac{8x}{11}\sqrt{11ax}$.

33. Reduce $a^2\sqrt[3]{\frac{16a}{3y^3}}$ to an integral Surd in its simplest form.

Ans. $\frac{2a^2}{3y}\sqrt[3]{18a}$.

34. Reduce $5x\sqrt{\frac{1+y^2}{36}}$ to an integral Surd in its simplest form.

Ans. $\frac{5x}{6}\sqrt{1+y^2}$.

35 Reduce $\frac{2}{5}\sqrt{\frac{18}{a+x}}$ to an integral Surd in its simplest form.

Ans. $\frac{6}{5(a+x)}\sqrt{2(a+x)}$

36. Reduce $5\sqrt{2}$ and $3\sqrt[3]{4}$ to Surds of the *same root.* (239).

By reducing the *exponents* of the Surd factor to the common denominator, we find

$$2^{\frac{1}{2}}=2^{\frac{3}{6}}=8^{\frac{1}{6}};\text{ and }4^{\frac{1}{3}}=4^{\frac{2}{6}}=16^{\frac{1}{6}}.$$

Ans. $5\sqrt[6]{8}$, and $3\sqrt[6]{16}$.

Prefixing the rational *co-efficients*, we have

$$5\sqrt{2}=5\sqrt[6]{8},\text{ and }3\sqrt[3]{4}=3\sqrt[6]{16}.$$

37. Reduce $2\sqrt[3]{5}$ and $5\sqrt{3}$ to Surds of the same root.

Ans. $2\sqrt[6]{25}$, and $5\sqrt[6]{27}$.

38. Reduce $a\sqrt{5}$ and $x\sqrt[3]{2}$ to Surds of the same root.

Ans. $a\sqrt[6]{125}$, and $x\sqrt[6]{4}$.

39. Reduce $10\sqrt{10}$ and $2\sqrt[4]{3x}$ to Surds of the same root.

Ans. $10\sqrt[4]{100}$, and $2\sqrt[4]{3x}$

40. Reduce $7\sqrt[3]{3y}$ and $2y\sqrt[4]{xy^2}$ to Surds of the same root·

Ans. $7\sqrt[12]{81y^4}$, and $2y\sqrt[12]{x^3y^6}$

41. Reduce $a^2x\sqrt[4]{\frac{3}{4}}$ and $\frac{3}{4}\sqrt{a^2x}$ to Surds of the same root.

Ans. $a^2x\sqrt[4]{\frac{3}{4}}$, and $\frac{3}{4}\sqrt[4]{a^4x^2}$.

42. Reduce $\frac{a}{2x}\sqrt[3]{2}$ and $5y\sqrt{\frac{1}{2x}}$ to Surds of the same root.

Ans. $\frac{a}{2x}\sqrt[6]{4}$, and $5y\sqrt[6]{\frac{1}{8x^3}}$

43. Reduce $2x\sqrt{a}$ and $\frac{1}{y}\sqrt[4]{\frac{1}{2}}$ to Surds of the same root.

Ans. $2x\sqrt[4]{a^2}$, and $\frac{1}{y}\sqrt[4]{\frac{1}{2}}$.

44. Reduce $\frac{2}{3}\sqrt[3]{10}$ and $31\sqrt{\frac{a}{y}}$ to Surds of the same root.

Ans. $\frac{2}{3}\sqrt[6]{100}$, and $31\sqrt[6]{\frac{a^3}{y^3}}$.

45. Reduce $\frac{1}{2y}\sqrt[3]{\frac{1}{a}}$ and $\frac{1}{2x}\sqrt[4]{\frac{1}{y^2}}$ to Surds of the same root.

Ans. $\frac{1}{2y}\sqrt[12]{\frac{1}{a^4}}$, and $\frac{1}{2x}\sqrt[12]{\frac{1}{y^6}}$.

ADDITION AND SUBTRACTION OF SURDS.

(240.) 1. The Sum, or Difference, of *similar surds* is obtained by prefixing the sum, or difference, of their *coefficients* as a coefficient to the common *radical factor.*

2. *Dissimilar surds* can be added together, or subtracted the one from the other, only by the proper *sign;* but Surds apparently dissimilar often become similar when reduced to their simplest forms, (237).

EXAMPLE.

To Add together $5\sqrt{80a}$ and $3\sqrt{125a}$.

Reducing the surds to their simplest forms, we find

$$5\sqrt{80a}=5\sqrt{16}\sqrt{5a}=20\sqrt{5a};$$
$$\text{and } 3\sqrt{125a}=3\sqrt{25}\sqrt{5a}=15\sqrt{5a}.$$

The two given Surds have thus become *similar,* (232.) Adding together the coefficients 20 and 15, we find the *Sum* $35\sqrt{5a}$.

The *Difference* of the two given Surds, is $(20-15)\sqrt{5a}=5\sqrt{5a}$.

The Addition or Subtraction of similar Surds is evidently nothing more than the addition or subtraction of similar *monomials;* thus

$$20 \text{ times} \sqrt{5a}+15 \text{ times } \sqrt{5a} \text{ is } 35 \text{ times } \sqrt{5a};$$
$$\text{just as } 20a+15a \text{ is } 35a.$$

EXERCISES

On the Addition and Subtraction of Surds.

1. Find the Sum of $3\sqrt{27}$ and $2\sqrt{48}$. *Ans.* $17\sqrt{3}$.
2. Find the Difference between $\sqrt{50}$ and $\sqrt{72}$. *Ans.* $\sqrt{2}$.
3. Find the Sum of $7\sqrt{28}$ and $6\sqrt{63}$. *Ans.* $32\sqrt{7}$.
4. Find the Difference between $21\sqrt{2}$ and $5\sqrt{18}$. *Ans.* $6\sqrt{2}$.
5. Find the Sum of $\sqrt{180}$ and $\sqrt{405}$. Ans. $15\sqrt{5}$.

6. Find the Difference between $\sqrt{18}$ and $2\sqrt{50}$. *Ans.* $7\sqrt{2}$.

7. Find the Sum of $\sqrt{12a^2}$ and $\sqrt{27a^2}$. *Ans.* $5a\sqrt{3}$.

8. Find the Difference between $3\sqrt{24x^2}$ and $\sqrt{54x^2}$. *Ans.* $3x\sqrt{6}$

9. Find the Sum of $4\sqrt{3a}$ and $\sqrt{48a}$. *Ans.* $8\sqrt{3a}$.

10. Find the Difference between $\sqrt{4a^3}$ and $\sqrt{9a^3}$. *Ans.* $a\sqrt{a}$.

11. Find the Sum of $3\sqrt[3]{4}$ and $7\sqrt[3]{4}$. *Ans.* $10\sqrt[3]{4}$

12. Find the Difference between $9\sqrt{200}$ and $\sqrt{288}$. *Ans.* $78\sqrt{2}$.

13 Find the Sum of $4\sqrt[3]{54}$ and $2\sqrt[3]{250}$. *Ans.* $22\sqrt[3]{2}$.

14. Find the Difference between $5\sqrt{9x^3}$ and $3\sqrt{x^3}$. *Ans.* $12x\sqrt{x}$.

15. Find the Sum of $2\sqrt[3]{16a}$ and $\sqrt[3]{54a}$. *Ans.* $7\sqrt[3]{2a}$.

16. Find the Difference between $3\sqrt[4]{10}$ and $5\sqrt[4]{10}$. *Ans.* $2\sqrt[4]{10}$

17. Find the Sum of $5\sqrt{98x}$ and $10\sqrt{2x}$. *Ans.* $45\sqrt{2x}$

18. Find the Difference between $3a\sqrt[3]{5}$ and $a\sqrt[3]{5}$. *Ans.* $2a\sqrt[3]{5}$.

19. Find the Sum of $a\sqrt[3]{y^2}$ and $3b\sqrt[3]{y^2}$. *Ans.* $(a+3b)\sqrt[3]{y^2}$.

20. Find the Difference between $5\sqrt[4]{5}$ and $2a\sqrt[4]{5}$. *Ans.* $(5-2a)\sqrt[4]{5}$

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21. Find the Sum of $(a+1)^{\frac{1}{2}}$ and $\sqrt{4a+4}$.

Ans. $3\,(a+1)^{\frac{1}{2}}$

22. Find the Difference between $\sqrt{1+x}$ and $3\,(1+x)^{\frac{1}{2}}$.

Ans. $2\,(1+x)^{\frac{1}{2}}$

23. Find the Sum of $2\,(a-x)^{\frac{1}{2}}$ and $\sqrt{9a-9x}$.

Ans. $5\,(a-x)^{\frac{1}{2}}$

24. Find the Difference between $\sqrt[3]{2+y}$ and $4\,(y+2)^{\frac{1}{3}}$.

Ans. $3\,(2+y)^{\frac{1}{3}}$

25. Find the Difference between $4\,(1+x^2)^{\frac{1}{2}}$ and $4\sqrt[3]{x^2+1}$.

Ans. $4\,(\overline{1+x^2}^{\frac{1}{2}}-\overline{x^2+1}^{\frac{1}{3}})$.

MULTIPLICATION AND DIVISION OF SURDS.

(241.) 1. The Product, or Quotient, of two Surds of the *same root*, is obtained by prefixing the product, or quotient, of their *coefficients* as a coefficient to the product, or quotient, of the *radical factors*,—the latter being affected with the same fractional *exponent* or radical sign.

2. Surds of *different roots* may be reduced to equivalent ones of the *same root*, (239), and then multiplied, or divided, as above. But

3. *Any two roots of the same quantity* may be multiplied into each other, by adding together their fractional *exponents;* or divided, the one into the other, by subtracting the exponent of the *divisor* from that of the dividend.

EXAMPLE.

To find the Product of $2\sqrt{10}\times 3\sqrt{2}$.

Since it is immaterial in what *order* the four factors are taken, we may take them in the order;

$2\times 3\sqrt{10}\times\sqrt{2}$; which gives the Product $6\sqrt{20}$, (234), $=12\sqrt{5}$, (237)

The *Quotient* of $2\sqrt{10}\div 3\sqrt{2}$ is $\frac{2}{3}\sqrt{5}$, since this quotient multiplied by the *divisor* produces the *dividend*.

EXERCISES

On the Multiplication and Division of Surds.

1.	Find the Product of $5\sqrt{8}\times 3\sqrt{5}$.	*Ans.* $30\sqrt{10}$.
2.	Find the Quotient of $6\sqrt{54}\div 3\sqrt{2}$.	*Ans.* $6\sqrt{3}$.
3.	Find the Product of $\sqrt{108}\times 2\sqrt{6}$.	*Ans.* $36\sqrt{2}$.
4.	Find the Quotient of $2\sqrt{96}\div\sqrt{54}$.	*Ans.* $2\frac{2}{3}$.
5.	Find the Product of $3\sqrt{5ax}\times 4\sqrt{20a}$.	*Ans.* $120a\sqrt{x}$.
6.	Find the Quotient of $4\sqrt{12a}\div 2\sqrt{6}$.	*Ans.* $2\sqrt{2a}$.
7.	Find the Product of $\sqrt{3ax}\times 3\sqrt{ax}$.	*Ans.* $3ax\sqrt{3}$
8.	Find the Quotient of $6\sqrt{12x^2}\div 3\sqrt{4}$.	*Ans.* $2x\sqrt{3}$.
9.	Find the Product of $\sqrt[3]{18}\times 5\sqrt[3]{4}$.	*Ans.* $10\sqrt[3]{9}$.
10.	Find the Quotient of $4\sqrt[3]{72}\div 2\sqrt[3]{18}$.	*Ans.* $2\sqrt[3]{4}$.

11. Find the Product of $5\sqrt{a} \times 3\sqrt[3]{a}$.

By reducing the surds to the *same root*, we obtain

$$5\sqrt[6]{a^3} \text{ and } 3\sqrt[6]{a^2}, \ (239).$$

These are to be multiplied together as before. *Ans.* $15\sqrt[6]{a^5}$.

But the given roots of the *same quantity* a, may also be multiplied into each other, by adding together their fractional *exponents* $\frac{1}{2}$ and $\frac{1}{3}$. Thus,

$$5a^{\frac{1}{2}} \times 3a^{\frac{1}{3}} = 15a^{\frac{5}{6}} = 15\sqrt[6]{a^5}, \text{ as before.}$$

Either of these two methods may be applied to roots of the same quantity. The first only is applicable to *different roots of different quantities.*

12. Find the Product of $3\sqrt[3]{x} \times 2\sqrt{x}$. *Ans.* $6\sqrt[6]{x^5}$.
13. Find the Quotient of $2\sqrt{3} \div 5\sqrt[3]{3}$. *Ans.* $\frac{2}{5}\sqrt[6]{3}$.
14. Find the Product of $4\sqrt{3} \times 2\sqrt[3]{2}$. *Ans.* $8\sqrt[6]{108}$
15. Find the Quotient of $8\sqrt[3]{a} \div 4\sqrt[4]{a}$. *Ans.* $2\sqrt[12]{a}$.
16. Find the Product of $\sqrt{ax} \times 3\sqrt[4]{ax}$. *Ans.* $3\sqrt[4]{a^3x^3}$.
17. Find the Quotient of $4\sqrt{x^3} \div 2\sqrt[4]{x}$. *Ans.* $2x\sqrt[4]{x}$.
18. Find the Product of $\frac{1}{2}\sqrt{5} \times \frac{1}{3}\sqrt[4]{10}$. *Ans.* $\frac{1}{6}\sqrt[4]{250}$.
19. Find the Quotient of $a^3\sqrt{\frac{1}{2}} \div a\sqrt[3]{\frac{1}{2}}$. *Ans.* $a^2\sqrt[6]{\frac{1}{2}}$.

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20. Find the Product of $(3+2\sqrt{5}) \times (2-\sqrt{5})$.

In cases of this kind, in which a Surd is connected with another quantity by the sign + or —, the Multiplication, or Division, must be performed as on *polynomials.*

$$\begin{array}{l} 3+2\sqrt{5} \\ \underline{2-\ \sqrt{5}} \\ 6+4\sqrt{5} \\ \underline{\quad -3\sqrt{5}-10} \\ 6+\ \sqrt{5}-10=\sqrt{5}-4. \end{array}$$

Each term of the *multiplicand* is multiplied by each term of the *multiplier*, and the partial products are then added together.

Observe that $2\sqrt{5} \times -\sqrt{5} = -2\sqrt{25} = -10$.

21. Find the Product of $(2+3\sqrt{2}) \times (1+5\sqrt{2})$.

Ans. $13\sqrt{2}+3$

22. Find the Product of $(4-\sqrt{3})\times(2+3\sqrt{3})$.

Ans. $10\sqrt{3}-1$.

23. Find the Product of $(5+2\sqrt{6})\times(1+2\sqrt{6})$.

Ans. $12\sqrt{6}+29$.

24. Find the Product of $(1-4\sqrt{7})\times(3-3\sqrt{7})$.

Ans. $87-15\sqrt{7}$.

25. Find the Quotient of $(\sqrt{20}+\sqrt{12}\div(\sqrt{5}+\sqrt{3})$.

Ans. 2.

RATIONALIZATION OF SURD DIVISORS.

(242.) In computing an approximate value of an *irrational* numerical expression, it is expedient that a *surd divisor* or denominator be made *rational*.

For example, suppose we wish to compute an approximate value of

$$\frac{2}{\sqrt{3}},\ \text{2 divided by the square root of 3.}$$

If we extract the square root of 3, for a divisor, a regard to *accuracy* will require that the root be continued to *several figures*, and hence will arise the inconvenience of *dividing by a large number*.

By multiplying both terms by the denominator, we have

$$\frac{2}{\sqrt{3}}=\frac{2\sqrt{3}}{\sqrt{9}}=\frac{\sqrt{12}}{3},\ \text{in which the divisor is } \textit{rational}.$$

The value will therefore be found by taking $\frac{1}{3}$ of the square root of 12; and by this method the computation is much simplified.

In pursuance of the object at present in view, it is necessary

To find a Multiplier of a given Surd which will cause the Product to be Rational.

(243.) 1. A *monomial* Surd will produce a rational quantity by being multiplied into itself with its *exponent subtracted from a unit.*

Thus $a^{\frac{1}{3}}$ multiplied by $a^{1-\frac{1}{3}}$, or $a^{\frac{1}{3}}\times a^{\frac{2}{3}}=a$, (241...3).

2. A *binomial* in which one or both terms contain an irrational *square root*, will produce a rational quantity by being multiplied into itself with a *sign changed.*

Thus $(\sqrt{3}+\sqrt{2})\times(\sqrt{3}-\sqrt{2})=3-2=1$.

The product in this case is readily found on the principle, that the Product of the *sum* and *difference* of two quantities is equal to the *difference of the squares* of the two quantities.

3. A *trinomial* containing irrational *square roots* will produce a *binomial* Surd by being multiplied into itself with a *sign changed*; and this binomial may be rationalized as above.

These principles provide for the most useful cases of the subject under consideration.—In applying them to the rationalization of surd denominators, both terms of the given Fraction must be multiplied by the same quantity, (81).

EXERCISES

On the Rationalization of Surd Denominators.

1. Reduce $\dfrac{2+\sqrt[3]{3}}{\sqrt[3]{3}}$ to a Fraction having a *rational* denominator.

Ans. $\dfrac{2\sqrt[3]{9}+3}{3}$.

2. Reduce $\dfrac{5}{3+\sqrt{5}}$ to a Fraction having a rational denominator.

Ans. $\dfrac{15-5\sqrt{5}}{4}$.

3. Reduce $\dfrac{2a}{2-\sqrt{7}}$ to a Fraction having a rational denominator.

Ans. $\dfrac{4a+2a\sqrt{7}}{-3}$.

4. Reduce $\dfrac{3x}{\sqrt{a}-\sqrt{x}}$ to a Fraction having a rational denominator.

Ans. $\dfrac{3x\sqrt{a}+3x\sqrt{x}}{a-x}$

5. Reduce $\dfrac{10}{a+\sqrt{10}}$ to a Fraction having a rational denominator.

Ans. $\dfrac{10a-10\sqrt{10}}{a^2-10}$.

6. Reduce $\dfrac{a-\sqrt[4]{2}}{\sqrt[4]{2}}$ to a Fraction having a rational denominator

Ans. $\dfrac{a\sqrt[4]{8}-2}{2}$.

7. Reduce $\dfrac{8}{\sqrt{3}+\sqrt{2}+1}$ to a Fraction having a rational denominator

Ans. $4+2\sqrt{2}-2\sqrt{6}$.

INVOLUTION AND EVOLUTION OF SURDS.

(244.) The Powers and Roots of *irrational* quantities are obtained, or *indicated*, according to the general principles of Involution and Evolution which have been established in the preceding Chapter.

We present here however a particular case of the

Square Root of Binomial Surds.

(245.) A Numerical Binomial of the form $a \pm \sqrt{b}$ admits of a square root in a *rational* and an *irrational* term, or two irrational terms, whenever $a^2 - b$ is a *perfect square.*

To determine the method to be pursued in this case of evolution, we must find

Formulas for the Square Root of $a \pm \sqrt{b}$.

The *square of the sum* of any two quantities, is equal to the *sum of their squares* + *twice their product*, (59). The binomial $a + \sqrt{b}$ may therefore represent the square of the sum of a *rational* and an *irrational* numerical term, or of two irrational terms, in the square root; a representing the *sum of the squares* of the two terms,—which sum will necessarily be *rational*,—and $\sqrt{b}$ representing *twice the product* of the two terms.

In like manner $a - \sqrt{b}$ may represent the square of the difference of a rational and an irrational numerical term, or of two irrational numerical terms, in the square root, (60).

If therefore we take x and y to represent the two terms of the square root of $a \pm \sqrt{b}$, we shall have,

$$(1) \qquad x^2 + y^2 = a\,;$$

$$(2) \qquad x + y = \sqrt{a + \sqrt{b}}\,;$$

$$(3) \qquad x - y = \sqrt{a - \sqrt{b}}\,;$$

Multiplying together equations (2) and (3), we have,

$$(4) \qquad x^2 - y^2 = \sqrt{a^2 - b}. \qquad (58.)$$

Adding together the (4) and (1), and also subtracting the (4) from the (1), and dividing by 2, we shall find,

$$(5)\qquad x^2=\frac{a+\sqrt{a^2-b}}{2};$$

$$(6)\qquad y^2=\frac{a-\sqrt{a^2-b}}{2}.$$

Extracting the square root of each of these equations,

$$\text{we find } x=\sqrt{\frac{a+\sqrt{a^2-b}}{2}};$$

$$\text{and } y=\sqrt{\frac{a-\sqrt{a^2-b}}{2}}.$$

Substituting these values of x and y in equations (2) and (3),—and interchanging the first and second members,—we have,

$$\text{(A)}\quad \sqrt{a+\sqrt{b}}=\sqrt{\frac{a+\sqrt{a^2-b}}{2}}+\sqrt{\frac{a-\sqrt{a^2-b}}{2}};$$

$$\text{(B)}\quad \sqrt{a-\sqrt{b}}=\sqrt{\frac{a+\sqrt{a^2-b}}{2}}-\sqrt{\frac{a-\sqrt{a^2-b}}{2}}.$$

These are the *Formulas* required. The right hand member of each will contain at most but two *irrational* terms, when (a^2-b) is a *perfect square*, that is, has an exact *square root.*

EXAMPLE.

To find the Square Root of $6+2\sqrt{5}$, or $6+\sqrt{20}$, (236.)

Substituting 6 for a, and $\sqrt{20}$ for $\sqrt{b}$, in Formula (A),

$$\sqrt{6+\sqrt{20}}=\sqrt{\frac{6+\sqrt{36-20}}{2}}+\sqrt{\frac{6-\sqrt{36-20}}{2}}.$$

And since $\sqrt{36-20}=\sqrt{16}=4$, the second member reduces to

$$\sqrt{\frac{6+4}{2}}+\sqrt{\frac{6-4}{2}}=\sqrt{5}+1, \text{ the } \textit{Root required.}$$

The root $\sqrt{5}+1$ may be verified by *squaring* it.

Thus $(\sqrt{5}+1)^2=5+2\sqrt{5}+1=6+2\sqrt{5}$.

Formula (B) would give the square root of $6-2\sqrt{5}$.

EXERCISES

On the Powers and Roots of Surds.

It will be readily perceived that by *cancelling the exponent or sign of a root, that root is raised to the corresponding power.*

Thus the *square* of $\sqrt{a}$ is a; the *cube* of $\sqrt[3]{a}$ is a, &c.

1. Find the *Square*, and also the square *root*, of $5\sqrt[3]{4}$.

The square of $5\times4^{\frac{1}{3}}$ is $25\times4^{\frac{2}{3}}=25\sqrt[3]{16}$, (200).
Reducing this result to its *simplest form*, we have

$$25\sqrt[3]{16}=25\sqrt[3]{8}\sqrt[3]{2}=50\sqrt[3]{2},\ (237).$$

The square *root* of $5\times4^{\frac{1}{3}}$ is $5^{\frac{1}{2}}\times4^{\frac{1}{6}}$, or $\sqrt{5}\sqrt[6]{4}$, (208).
Multiplying together the *two factors* of this root, we have

$$\sqrt{5}\sqrt[6]{4}=\sqrt[6]{125}\sqrt[6]{4}=\sqrt[6]{500},\ (241\ldots2).$$

2. Find the Square, and also the square *root*, of $9\sqrt[3]{3}$.
Ans. $81\sqrt[3]{9}$, and $3\sqrt[6]{3}$.

3. Find the Cube, and also the cube *root*, of $3\sqrt{2}$.
Ans. $54\sqrt{2}$, and $\sqrt[6]{18}$.

4. Find the Square, and also the square *root*, of $2\sqrt[3]{3}$.
Ans. $4\sqrt[3]{9}$, and $\sqrt[6]{24}$.

5. Find the Cube, and also the cube *root*, of $3a\sqrt{3}$.
Ans. $81a^3\sqrt{3}$, and $\sqrt[6]{27a^2}$.

6. Find the Square, and also the square *root*, of $x^2\sqrt[3]{5}$.
Ans. $x^4\sqrt[3]{25}$, and $x\sqrt[6]{5}$.

7. Find the Square, and also the square *root*, of $3\sqrt{a+1}$.
Ans. $9(d+1)$, and $\sqrt[4]{9(a+1)}$.

8. Find the Square, and also the square *root*, of $4\sqrt[3]{1-x}$.
Ans. $16\sqrt[3]{(1-x)^2}$, and $2\sqrt[6]{1-x}$

9. Find the Cube, and also the cube *root*, of $2\sqrt{a^2-x}$.
Ans. $8\sqrt{(a^2-x)^3}$, and $\sqrt[6]{4(a^2-x)}$.

10. Find the Square, and also the square *root*, of $3\sqrt{2-a}$.
Ans. $9(2-a)$, and $\sqrt[4]{9(2-a)}$.

11. Find the Square of $2+\sqrt{3}$, and of $2-\sqrt{3}$.

These squares may be found by multiplying each binomial into itself, or, more readily, by applying the propositions relating to the squares of the *sum* and the *difference* of two quantities.

Ans. $7+4\sqrt{3}$, and $7-4\sqrt{3}$.

12. Find the square of $3+2\sqrt{5}$, and of $5-3\sqrt{2}$.

Ans. $12\sqrt{5}+29$, and $43-30\sqrt{2}$.

13. Find the Square of $\sqrt{5+2\sqrt{a}}$, and of $\sqrt{3-5\sqrt{x}}$.

Ans. $5+2\sqrt{a}$, and $3-5\sqrt{x}$.

14. Find the Square of $\sqrt{2}+3\sqrt{5}$, and of $\sqrt{a}-2x\sqrt{x}$.

Ans. $6\sqrt{10}+47$, and $a-4x\sqrt{ax}+4x^3$.

15. Find the Cube of $\sqrt[3]{1+5\sqrt{4}}$, and of $\sqrt[3]{a-3\sqrt{a}}$.

Ans. $1+5\sqrt{4}$, and $a-3\sqrt{a}$.

16. Find the Square of $\sqrt{3}+a\sqrt{3}$, and of $\sqrt{3}-3\sqrt{a}$.

Ans. $3+6a+3a^2$, and $3-6\sqrt{3a}+9a$.

17. Find the Cube of $2+2\sqrt{a}$, and of $2-a\sqrt{2}$.

Ans. $8+24a+(24+8a)\sqrt{a}$, and $8+12a^2-(12a+2a^3)\sqrt{2}$.

18. Find the Cube of $(\sqrt{a}+\sqrt{2})^{\frac{1}{3}}$ and of $(\sqrt[3]{1}-\sqrt{y})^{\frac{1}{3}}$.

Ans. $\sqrt{a}+\sqrt{2}$, and $\sqrt[3]{1-\sqrt{y}}$.

19. Find the Square *root* of $a+2\sqrt{ax}+x$, (218).

Ans. $\sqrt{a}+\sqrt{x}$.

20. Find the Square root of $a+2\sqrt{a}+1$, and of $3+2a\sqrt{3}+a^2$.

Ans. $\sqrt{a}+1$, and $\sqrt{3}+a$.

21. Find the Square root of $x-2\sqrt{x}+1$, and of $5-2x\sqrt{5}+x^2$.

Ans. $\sqrt{x}-1$, and $\sqrt{5}-x$.

22. Find the Square root of $23+8\sqrt{7}$, (245). *Formula A.*

Ans. $4+\sqrt{7}$.

23. Find the Square root of $19+8\sqrt{3}$, and of $7-2\sqrt{10}$.

Ans. $4+\sqrt{3}$, and $\sqrt{5}-\sqrt{2}$.

IMAGINARY QUANTITIES.

(246.) An *even root* of a *negative* quantity is *impossible*, and the symbol of such a root is therefore called an *imaginary* or *impossible* quantity, in contradistinction to *real* quantities.

Thus $\sqrt{-4}$, the square root of -4, is *imaginary*, since there is no quantity whose *square* is -4; and for a like reason $\sqrt{-16}$ is *imaginary*.

An Imaginary Quantity results in calculation from some *impossibility* in the conditions of a Problem, and may therefore be regarded as a sign of such impossibility.

As an instance of this, suppose it were required *To find a number whose square subtracted from* 5 *shall leave* 9.

If x represent the required number, the Equation of the problem will be

$$5-x^2=9.$$

From this equation, $x^2=-4$; and hence $x=\sqrt{-4}$

The value of x being *imaginary* or impossible, shows that the *problem is impossible;* that is, that there is no number whose square subtracted from 5 will leave 9.

Imaginary Quantities may become the subjects of calculation like real quantities. Thus we may wish to *verify* an imaginary value of the unknown quantity in an Equation, by subjecting this value to all the operations which are performed on its *symbol* in the equation.

Calculus of Imaginary Quantities.

(247.) All the principles which have been established for the calculus of *radical* quantities are applicable to *imaginary* quantities, except that the ambiguous *sign* ± does not precede the *product* of two imaginary quantities, as is the case, in general, with *even roots of real quantities*.

The *sign* which affects the Product of two *imaginaries* may always be determined by means of imaginary and real *factors* into which such quantities may be resolved.

(248.) An Imaginary Quantity may always be resolved into the *like root* of (-1) multiplied into the like root of a *positive* quantity which is equal to the negative quantity in the given imaginary.

Thus -4 being equal to -1×4, we have $\sqrt{-4}=\sqrt{-1}\times\sqrt{4}$;
and $-a$ being equal to $-1\times a$, we have $\sqrt{-a}=\sqrt{-1}\times\sqrt{a}$, (234.)

By means of this transformation it may be shown, that

(249.) The Product of two imaginary *square roots* is the *negative* square root of the product of the two quantities under the $\sqrt{\ }$, *if the given roots are preceded by like signs*, + or − ; otherwise, it is the *positive* square root of that product.

For example, $\sqrt{-a}\ .\ \sqrt{-b}=-\sqrt{ab}$;
and, $\sqrt{-a}\ .\ -\sqrt{-b}=+\sqrt{ab}$.

On general principles of calculation, the Product of the square roots of $-a$ and $-b$, is equal to the square root of the product ab, which would be $+\sqrt{ab}$, or else $-\sqrt{ab}$, (215).

But the ambiguity in the *sign* to be prefixed to an *even root* in general, is removed *when we know the factors which entered into the composition of the quantity* whose root is considered. When a^2, for example, is known to have been derived from $a\times a$, the square root of a^2 is a, and not $-a$.

To determine the *sign* of $\sqrt{ab}$ in the first example, we have

$$\begin{aligned}\sqrt{-a}\ .\ \sqrt{-b}&=\sqrt{-1}\ .\ \sqrt{a}\ .\ \sqrt{-1}\ .\ \sqrt{b},\ (248)\\&=\sqrt{-1}\ .\ \sqrt{-1}\ .\ \sqrt{a}\sqrt{b}\\&=(\sqrt{-1})^2\sqrt{ab}=-1\sqrt{ab},\text{ or }-\sqrt{ab}.\end{aligned}$$

In the second example, we have

$$\begin{aligned}\sqrt{-a}\ .\ -\sqrt{-b}&=\sqrt{-1}\ .\ \sqrt{a}\ .\ -\sqrt{-1}\ .\ \sqrt{b},\ (248),\\&=\sqrt{-1}\ .\ -\sqrt{-1}\ .\ \sqrt{a}\sqrt{b}\\&=-(\sqrt{-1})^2\sqrt{ab}=-(-1)\sqrt{ab}=+\sqrt{ab}.\end{aligned}$$

In this example the square $(\sqrt{-1})^2$ has the sign − before it, because it results from multiplying $\sqrt{-1}$ by $-\sqrt{-1}$, (42); and $-(-1)$ becomes $+1$ by changing the sign in subtracting.

CHAPTER X.

QUADRATIC AND OTHER EQUATIONS.

(250.) A QUADRATIC EQUATION, or an equation of the *second degree*, is one in which the highest power of the unknown quantity is its *square*, or second power, as

$$3x^2=12;\ \text{or}\ 3x^2+4x=20.$$

A *Cubic Equation*, or an equation of the *third degree*, is one in which the highest power of the unknown quantity is its *cube*, or third power; and in like manner is defined a *Biquadratic* Equation, or an equation of the fourth degree, and so on.

An Equation containing two or more unknown quantities is of the *degree* which corresponds to the greatest number of *unknown factors* in any of its terms.

Thus $3xy+y=a$ is an Equation of the *second degree*, its first term containing the unknown factors xy.

And $x^2y-y=b$ is an equation of the third degree, its first term containing three unknown factors xxy.

Pure and Affected Equations.

(251.) A *Pure Equation* is an equation which contains but *one power* of the unknown quantity; and is a Simple Equation, a Pure Quadratic, or a Pure Cubic, &c., according to its degree.

Thus $3x^2$ is a pure *quadratic;* $x^3=64$ is a pure *cubic.*

An *Affected Equation* is one which contains different powers of the unknown quantity. When these powers are in regular *ascension*, beginning with the first power, the Equation is also called a *complete* equation.

Thus $x^2+3x=10$ is an *affected*, and also a *complete* quadratic; $x^3-2x^2=75$, or $x^3-3x=110$, is an *affected cubic;* and $x^3+3x^2+4x=28$ is a complete cubic.

Roots of Equations.

(252.) A *Root* of an Equation is a value of the unknown quantity in the equation. It will presently be shown that an equation of the 2d degree has *two roots*, of the 3d degree *three* roots, and so on.

In the Simple Equation $3x=15$, the value of the unknown quantity x is 5; then 5 is the root of the equation.

It is evident that in a Simple Equation there can be but *one* value of the unknown quantity that will *satisfy the equation*. An equation of the 1st degree has therefore but *one root*.

GENERAL PROPERTIES OF EQUATIONS.

1. *Divisors of an Equation.*

(253.) If a be a *root* of an Equation of any degree, containing but one unknown quantity, x; the equation—with all its terms *transposed to one side*—will be divisible by $x-a$.

Let a be a root of the Equation

$$x^2+mx=s;$$

then will the equation $x^2+mx-s=0$ be divisible by $x-a$

For let r be the *remainder*, if any, after the quotient q has been obtained; then will

$$x^2+mx-s=(x-a)q+r=0;$$

the dividend being equal to the remainder added to the product of the *divisor* and *quotient*.

But a being a value of x, (252), we have $x-a=0$; then the $(x-a)\times q$ is 0, (43); and consequently $r=0$; that is, the division will leave *no remainder*.

The preceding demonstration is applicable to an equation of the third, fourth, or any higher degree.

(254.) Conversely, If an Equation of any degree, containing but one unknown quantity, x,—with all its terms transposed to one side—be divisible by $x-a$; then a will be a *root* of the equation.

This is evident from considering that the divisibility of the Equation, as shown above, depends on the condition that $x-a=0$, or that a is a value of x.

2. *Number of Roots of an Equation.*

(255.) Every Equation containing but one unknown quantity, has just as many *roots* as there are units in the exponent of the highest power of the unknown quantity in the equation.

Let a represent a root of the *cubic* Equation

$$x^3+mx^2+nx=s.$$

Transposing s to the first side of the equation, we have

$$x^3+mx^2+nx-s=0.$$

Dividing this equation by $x-a$, (253), we shall obtain an equation of the *second degree*, which may be represented by

$$x^2+px-q=0.$$

Let b be a root of this equation. Dividing the equation by $x-b$ we shall obtain an equation of the *first degree*, represented by

$$x-u=0.$$

The binomials $x-a$, $x-b$, and $x-u$, which represent the two *divisors* and the last *quotient*, are the factors of the dividend.

$$x^3+mx^2+nx-s.$$

The original equation is thus resolved, representatively, into

$$(x-a)(x-b)(x-u)=0.$$

Since this equation is divisible by each of these *three binomial factors*, it follows that a, b, and u are *three roots* of the equation, (254).

And since the given equation—being of the 3d degree—cannot be resolved into more than *three* binomial factors, each containing the *first* power of x, it cannot have more than *three roots.*

The same method of demonstration will show the proposition to be true for an Equation of any other degree.

The several *roots* of an Equation are not necessarily *unequal,* though such will usually be found to be the case. The preceding Proposition shows that an Equation may be resolved into binomial factors, each containing a *root* of the equation. Two or more of these roots may be equal to each other.

In the Equation $x^2-5x+6=(x-2)(x-3)=0$,
the two roots are 2 and 3.

In the Equation $x^3-7x^2+16x-12=(x-2)(x-2)(x-3)=0$
the three roots are 2, 2, and 3.

SOLUTION OF PURE EQUATIONS OF THE SECOND AND HIGHER DEGREES.

The Equations belonging to this class are those which, in their simplest forms, contain but one power of the unknown quantity

RULE XXII.

(256.) *For the Solution of a Pure Equation.*

1. Reduce the Equation to the form
$x^n = s$; in which x^n must be *positive.*

2. Extract that *root* of both sides of the equation which **corresponds** to the *power* of the unknown quantity

EXAMPLE.

To find the value of x in the Equation

$$\frac{x^2}{4} + 5 = 2x^2 - 23.$$

Clearing the equation of its Fraction, we have

$$x^2 + 20 = 8x^2 - 92.$$

Transposing, and adding similar terms, we find

$$-7x^2 = -112.$$

Dividing both sides of this equation by the coefficient -7,

$$x^2 = 16.$$

Extracting the *square root* of both sides,

$$x = \pm 4, \ (215).$$

The *two values* of x are thus found to be 4 and -4, (255), either of which will satisfy the given Equation.

It is evident that, in a Pure Equation of the *second degree*, the two values of the unknown quantity will always be *equal*, with *contrary signs.*

The unknown quantity may enter an Equation in a *surd* expression, which it will be necessary to *rationalize* in the solution of the equation.

Thus in the equation $\sqrt{x} + \sqrt{1+x} = a$, it would be necessary to *rationalize* x, that is, to clear it of the *radical sign* before the value of x could be determined.

The following observations will assist the student in the

Rationalization of Surd Quantities in an Equation.

(257.) A Surd quantity in an Equation will be *rationalized* by transposing all the other terms to the other side of the equation, and raising both sides to the *power* corresponding to the indicated *root.*

To rationalize x in the Equation

$$\sqrt{x+1}-a=b.$$

By transposition, $\sqrt{x+1}=a+b$

Squaring both sides, $x+1=a^2+2ab+b^2$.

(258.) Two Surds in an Equation may be rationalized by successive involutions,—in the first of which it will generally be expedient to make one of the Surds stand alone on one side of the equation.

To rationalize x in the Equation

$$\sqrt{x}+\sqrt{1+x}=a.$$

By transposition, $\sqrt{x}=a-\sqrt{1+x}$.

Squaring both sides, $x=a^2-2a\sqrt{1+x}+1+x$.

By transposition, $2a\sqrt{1+x}=a^2+1$.

Squaring both sides, $4a^2(1+x)=a^4+2a^2+1$.

(259.) When an Equation contains a Fraction whose terms are both *irrational,* it will sometimes be expedient to rationalize its *denominator* before clearing the equation of the fraction.

To rationalize x in the Equation

$$\frac{\sqrt{x}}{1+\sqrt{x}}=a.$$

Multiplying both terms of the Fraction by $1-\sqrt{x}$, (243...2),

$$\frac{\sqrt{x}-x}{1-x}=a.$$

Clearing this equation of its Fraction, and transposing, we have

$$\sqrt{x}=a-ax+x.$$

The Surd in this equation will be rationalized by *squaring both sides*, as in the preceding examples.

By the preceding methods we may generally rationalize one or more Surds containing the unknown quantity in an Equation. Other expedients, however, such as the *extraction of roots* in possible cases, in the course of the operation, will sometimes be requisite; but these must be left to the care and skill of the computer.

EXERCISES

On Rationalization, and Pure Equations.

1. Find the value of x in the equation

$$24-\sqrt{2x^2+9}=15. \qquad \textit{Ans. } x=\pm 6$$

2. Find the value of x in the equation

$$13-\sqrt{3x^2+16}=5. \qquad \textit{Ans. } x=\pm 4$$

3. Find the value of x in the equation

$$35+\sqrt[3]{x-5}=40. \qquad \textit{Ans. } x=130$$

4. Find the value of x in the equation

$$1+2\sqrt{x}=\sqrt{4x+21}. \qquad \textit{Ans. } x=25$$

5. Find the value of x in the equation

$$\sqrt{x-32}=\sqrt{x}-\tfrac{1}{2}\sqrt{32}. \qquad \textit{Ans. } x=50.$$

6. Find the value of x in the equation

$$3+\sqrt{x+4}\times\sqrt{x-4}=10. \qquad \textit{Ans. } x=\pm\sqrt{65}.$$

7. Find the value of x in the equation

$$a+\sqrt{x-3}\times\sqrt{x+3}=4a.$$

Ans. $x=\pm 3\sqrt{a^2+1}.$

8. Find the value of x in the equation

$$\sqrt{x}+\sqrt{5+x}=\frac{10}{\sqrt{5+x}}. \qquad \textit{Ans. } x=1\tfrac{2}{3}$$

9. Find the value of x in the equation

$$\frac{4\sqrt{6x}-9}{4\sqrt{6x}+6}=\frac{\sqrt{6x}-2}{\sqrt{6x}+2}. \qquad \textit{Ans. } x=6$$

10. Find the value of x in the equation

$$\frac{\sqrt{x+c}+\sqrt{x-c}}{\sqrt{x+c}-\sqrt{x-c}}=b. \qquad \textit{Ans. } x=\frac{cb^2+c}{2b}.$$

SOLUTION OF COMPLETE EQUATIONS OF THE SECOND DEGREE.

These Equations in their simplest forms contain no other power of the unknown quantity than its *square* and first power.

The value of the unknown quantity will be found by the following

RULE XXIII.

(260.) *For the Solution of a Complete Equation of the Second Degree.*

1. Reduce the Equation to the form
$x^2+bx=s$; in which x^2 must be *positive.*

2. Add the *square* of half the coefficient of x, in the second term, to each side of the equation:—the first side will then be a *perfect square.*

3. Extract the square root of each side, and the result will be a Simple Equation,—from which the value of x may readily be found.

EXAMPLE.

To find the value of x in the equation

$$2x+\frac{3x-5}{2}=5x-\frac{5x-5}{x-3}.$$

Clearing the equation of its fractions, transposing, and adding similar terms, we shall find

$$-3x^2+14x=-5.$$

Dividing both sides of this equation by -3,

$$x^2-\frac{14x}{3}=\frac{5}{3}.$$

Adding the *square of half the coefficient* of x, in the second term, to each side,—which is called *completing the square,*

$$x^2-\frac{14x}{3}+\frac{49}{9}=\frac{5}{3}+\frac{49}{9}=\frac{64}{9}.$$

Extracting the *square root* of each side,

$$x-\frac{7}{3}=\pm\frac{8}{3}.$$

Whence $x=\frac{7}{3}+\frac{8}{3}=5$; or $x=\frac{7}{3}-\frac{8}{3}=-\frac{1}{3}$. (255).

Either of these two values of x, 5 or $-\frac{1}{3}$, will satisfy the given equation; and each of the binomials $x-5$ and $x+\frac{1}{3}$ will divide the equation

$$x^2-\frac{14x}{3}-\frac{5}{3}=0. \quad (253).$$

By performing the division it will be found that the left hand side of this equation, is the *product of the two binomials*; that is,

$$(x-5)(x+\tfrac{1}{3})=x^2-\frac{14x}{3}-\frac{5}{3}=0.$$

On the preceding Rule we remark,

1. The method of *completing the square* in the first member, results from the composition of the *square* of a Binomial.

Thus the square of the binomial $a+b$ is $a^2+2ba+b^2$, in which b is half the coefficient of a in the second term.

2. If in reducing the equation to the required form, the first term should become $-x^2$, the signs of all the terms in the equation must be changed (117), before completing the square,—otherwise the root of the first side would be *imaginary*, (246).

3. The *square root* of the first side of the equation—after the completion of the square—will always be the square root of the *first term*, + or − *half the coefficient* of x in the second term, according as the second term is + or −.

Hence, *without completing the square on the first side*, we may shorten the solution, by observing that

4. In an equation of the form $x^2+bx=s$, the value of x is half the co-efficient of x in the second term, taken with a *contrary sign*, $\pm$ the square root of (the *second member* of the equation + the square of said half co-efficient).

Thus from the equation $x^2-\frac{14x}{3}=\frac{5}{3}$, we immediately find

$$x=\frac{7}{3}\pm\sqrt{\frac{5}{3}+\frac{49}{9}}=\frac{7}{3}\pm\sqrt{\frac{64}{9}}=\frac{7}{3}\pm\frac{8}{3}$$

EXERCISES

On Quadratic Equations with One Unknown Quantity.

1. Find the value of x in the equation
$x^2-15=45-4x$. *Ans.* $x=6$ or -10.

2. Find the value of x in the equation
$x^2+10=65+6x$. *Ans.* $x=11$ or -5.

3. Find the value of x in the equation
$2x^2+8x-30 \quad 60$. *Ans.* $x=5$ or -9.

4. Find the value of x in the equation
$3x^2-3x+9=8\frac{1}{3}$. *Ans.* $x=\frac{1}{3}$ or $\frac{2}{3}$.

5. Find the value of in the equation
$5x^2+4x-90=114$. *Ans.* $x=6$ or $-6\frac{4}{5}$.

6. Find the value of x in the equation
$\frac{1}{2}x^2-\frac{1}{4}x+2=9$. *Ans.* $x=4$ or $-3\frac{1}{2}$.

7. Find the value of x in the equation
$2x^2+12x+36=356$. *Ans.* $x=10$ or -16.

8. Find the value of x in the equation
$4x-46=\dfrac{36-x}{x}$. *Ans.* $x=12$ or $-\frac{3}{4}$.

9. Find the value of x in the equation
$8x^2+6=7x+171$. *Ans.* $x=5$ or $-4\frac{1}{8}$.

10. Find the value of x in the equation
$5x-23=\dfrac{25-3x}{x}$. *Ans.* $x=5$ or -1.

11. Find the value of x in the equation
$3x^2+6=3x+5\frac{1}{3}$. *Ans.* $x=\frac{1}{3}$ or $\frac{2}{3}$.

12. Find the value of x in the equation
$\dfrac{x^2}{2}-\dfrac{x}{3}+20\frac{1}{2}=42\frac{2}{3}$. *Ans.* $x=7$ or $-6\frac{1}{3}$.

13. Find the value of x in the equation
$\dfrac{8-x}{2}=\dfrac{x-2}{6}+\dfrac{2x-11}{x-3}$. *Ans.* $x=6$ or $\frac{1}{2}$

14. Find the value of x in the equation
$x+4=13-\dfrac{7x-8}{x}$. *Ans.* $x=4$ or -2

15 Find the value of x in the equation
$\dfrac{x-3a}{b}=\dfrac{9(b-a)}{x}$. *Ans.* $x=3b$, or $3(a-b)$

Another Method of Solving Quadratic Equations.

(261.) A Binomial of the form ax^2+bx will be made a *perfect square* by multiplying it by 4 *times* the coefficient of x^2, and adding the square of the given coefficient of x.

Thus $(ax^2+bx)4a+b^2=4a^2x^2+4abx+b^2=(2ax+b)^2$, (59).

To apply this principle to solution of the Equation

$$3x^2-2x=65.$$

Multiplying both sides of the equation by 4 *times* the coefficient 3, and adding the square of the coefficient 2 to both sides, we have

$$36x^2-24x+4=780+4=784.$$

Extracting the square root of both sides, we find

$$6x-2=\pm 28\text{; which gives }\quad x=\frac{\pm 28+2}{6}=5\text{ or }-4\tfrac{1}{3}.$$

The *square root of the first side* of the Equation prepared as above, will be x multiplied into *twice* the given coefficient of x^2, + or − the given coefficient of x in the second term, according as this term is + or −. This is evident from the preceding illustration, (261.)

From these principles we derive

RULE XXIV.

(262.) *To Reduce an Equation of the form* $ax^2+bx=s$ *to a Simple Equation.*

1. Double the coefficient of x^2, and divide the first member by x.

2. Multiply the second member by 4 *times* the given coefficient of x^2, add the square of the given coefficient of x, and extract the square root of the sum.

Applying this Rule to the numerical equation

$$3x^2-5x=50,$$

we immediately obtain $\quad 6x-5=\pm\sqrt{50\times 12+25}=\pm\sqrt{625}.$

When the coefficient of x^2 is *unity*, the multiplier of the second member will be simply 4; and when the coefficient of x is *unity*, the square to be added after multiplying the second member is 1.

This method of solving a Quadratic is preferable to the one first given, whenever the coefficient of x would give rise to a *fraction* in dividing the Equation by the coefficient of x^2, or in *completing the square*, according to that method.

16. Find the value of x in the equation
$3x^2+2x-9=76$. *Ans.* $x=5$, or $-5\frac{2}{3}$.

17. Find the value of x in the equation
$2x^2-14x+2=18$. *Ans.* $x=8$, or -1.

18. Find the value of x in the equation
$x^2-12x+50=0$. *Ans.* $x=6\pm\sqrt{-14}$.

19. Find the value of x in the equation
$\frac{1}{2}x^2-12\frac{1}{2}=\frac{1}{2}x+15\frac{1}{2}$. *Ans.* $x=8$, or -7.

20. Find the value of x in the equation
$3x^2-\frac{3}{4}=5\frac{1}{4}+2x$. *Ans.* $x=\frac{1}{3}\pm\frac{1}{3}\sqrt{19}$.

21 Find the value of x in the equation
$x^2-\frac{5x}{6}+\frac{1}{6}=0$. *Ans.* $x=\frac{1}{2}$, or $\frac{1}{3}$.

22. Find the value of x in the equation
$x+4=13-\frac{7x-8}{x}$. *Ans.* $x=4$, or -2.

23. Find the value of x in the equation
$\frac{x^2}{3}+\frac{4x}{5}-34\frac{1}{5}=0$. *Ans.* $x=9$, or $-11\frac{2}{5}$.

24. Find the value of x in the equation
$x^2-\frac{x}{2}=\frac{2}{3}-\frac{x}{3}$. *Ans.* $x=\frac{1}{12}\pm\frac{1}{12}\sqrt{97}$.

25. Find the value of x in the equation
$\frac{10}{x}-\frac{14-2x}{x^2}=2\frac{4}{9}$. *Ans.* $x=3$, or $1\frac{10}{11}$.

26. Find the value of x in the equation
$-\frac{x^2}{4}+3x+3=13$. *Ans.* $x=6\pm\sqrt{-4}$.

27. Find the value of x in the equation
$x^2+\frac{17x}{4}=-\frac{17x}{4}-4$. *Ans.* $x=-\frac{1}{2}$, or -8

28. Find the value of x in the equation,
$5x-\frac{3x-3}{x-3}=2x+\frac{3x-6}{2}$. *Ans.* $x=4$, or -1.

29. Find the value of x in the equation
$\frac{x+4}{3}-\frac{7-x}{x-3}=-1+\frac{4x+7}{9}$. *Ans.* $x=21$, or 5

30. Find the value of x in the equation
$\frac{x+a}{x}-\frac{3x}{x-a}=0$. *Ans.* $x=\pm a\sqrt{-\frac{1}{2}}$.

Equations in which the Unknown Quantity is contained in a Surd Expression.

31. Find the value of x in the equation

$$5+\sqrt{x^3+36}=15.$$

By transposition, $\sqrt{x^3+36}=15-5=10.$

Squaring both sides, $x^3+36=100$; (257).

from which, $x^3=64.$

We have now a *pure cubic* equation, in which x has necessarily *three* values or *roots*, (255).

Extracting the *cube root* of each side of the last equation, we find

$x=4$, which is one value of x.

To find the other two values of x, we must reduce the *cubic* equation to a *quadratic* by division, (253).

Dividing each side of the equation $x^3-64=0$ by $x-4$, we find

$x^2+4x+16=0$, or $x^2+4x=-16$; hence $x=-2\pm\sqrt{-12}.$

We have thus found $x=4$, or $-2+\sqrt{-12}$, or $-2-\sqrt{-12}$; the first value being *real*, the other two *imaginary*.

Each of these imaginary values of x, as well as the *real* value 4, will satisfy the equation $x^3=64.$

Thus x being equal to $-2+\sqrt{-12}$, we have, (247), (249),

$$x^2=(-2+\sqrt{-12})^2=4-4\sqrt{-12}-12=-8-4\sqrt{-12}.$$

and $$x^3=(-8-4\sqrt{-12})(-2+\sqrt{-12})=16+48=64,$$

In like manner the other imaginary value of x may be verified.

32. Find the value of x in the equation

$6+\sqrt{3x+4}=11.$ *Ans.* $x=7.$

33. Find the value of x in the equation

$24-\sqrt{2x^2+9}=15.$ *Ans.* $x=\pm 6.$

34. Find the value of x in the equation

$20-\sqrt{x^3+40}=4$

Ans. $x=6$ or $-3+\sqrt{-27}$, *or* $-3-\sqrt{-27}$.

35. Find the value of x in the equation

$$x+\sqrt{x}=20.$$

From this equation we shall find $x=25$ or 16. The value 25 will not satisfy the original equation if the square root of x be restricted to its *positive* value; but this root is $\pm\sqrt{x}$, and 25 satisfies the equation for the *negative* root.

Two values of x may be found in each of the next three Exercises; but only that value is given in the *Ans.* which corresponds to *positive* roots in the given Equation.

36. Find the value of x in the equation

$$2\sqrt{x}+\sqrt{x+9}=13. \qquad \textit{Ans. } x=16.$$

37. Find the value of x in the equation

$$2\sqrt{x}+\sqrt{2x+1}=\frac{21}{\sqrt{(2x+1)}}. \qquad \textit{Ans. } x=4.$$

38. Find the value of x in the equation

$$4\sqrt{x+16}=7\sqrt{x+16}-x-6. \qquad \textit{Ans. } x=9$$

39. Find the value of x in the equation

$$\frac{x+4}{2}=\sqrt{4+\sqrt{2x^3+x^2}}. \qquad \textit{Ans. } x=12, \text{ or } 4.$$

40. Find the value of x in the equation

$$(x-2)^2=\frac{x+\sqrt{(x^2-9)}}{x-\sqrt{(x^2-9)}}.$$

By *rationalizing* the denominator, (243...2), we shall find

$$(x-2)^2=\frac{(x+\sqrt{(x^2-9)})^2}{9}.$$

Extracting the square root of each side of this equation, we have

$$x-2=\frac{x+\sqrt{(x^2-9)}}{3}. \qquad \textit{Ans. } x=5, \text{ or } 3$$

41. Find the value of x in the equation

$$x^2-6x+9=\frac{x+\sqrt{(x^2-16)}}{x-\sqrt{(x^2-16)}}. \qquad \textit{Ans. } x=5. \text{ or } 4$$

42. Find the value of x in the equation

$$\sqrt[3]{x^3+37}\times(x^3+37)^{\frac{2}{3}}=64.$$

$$\textit{Ans. } x=3, \text{ or} -\tfrac{3}{2}\pm\tfrac{3}{2}\sqrt{-3}.$$

Equations of a Quadratic Form with reference to a Power or Root of the Unknown Quantity.

(263.) Any Equation containing the unknown quantity x in but two terms—with its exponent in one *double* its exponent in the other—is a Quadratic with reference to the *lower power* of x; and the value of such power may be found accordingly.

To find the value of x in the equation

$$x^{\frac{1}{2}}+4x^{\frac{1}{4}}=21.$$

The higher *fractional* power $x^{\frac{1}{2}}$ is the *square* of the lower $x^{\frac{1}{4}}$; and the equation is therefore *quadratic* with reference to $x^{\frac{1}{4}}$.

Completing the square, we have

$$x^{\frac{1}{2}}+4x^{\frac{1}{4}}+4=21+4=25.$$

Extracting the square root of each side,

$$x^{\frac{1}{4}}+2=\pm 5\,;$$

from which $x^{\frac{1}{4}}=3$, or -7.

By raising each of these values to the *4th power*, we find

$$x=81, \text{ or } 2401.$$

The first of these two values of x is easily verified. In verifying the value 2401 it must be observed that its 4th root is -7, and that $4x^{\frac{1}{4}}$ is therefore -28.

When x is in a *fractional* power, in the following Exercises, only that value will be given in the *Ans.* which satisfies the given form of the Equation.—Imaginary values of x are also omitted.

43. Find the value of x in the equation
$x^4-2x^2+6=230.$ *Ans.* $x=\pm 4.$

44. Find the value of x in the equation
$x^6+20x^3-10=59.$ *Ans.* $x=\sqrt[3]{3}.$

45. Find the value of x in the equation
$2x^4-x^2+20=23.$ *Ans.* $x=\frac{1}{2}\sqrt{6}.$

46. Find the value of x in the equation
$3x^{\frac{1}{2}}-5x^{\frac{1}{4}}=-1\frac{1}{3}.$ *Ans.* $x=3\frac{13}{81}.$

47. Find the value of x in the equation
$6x^{\frac{4}{3}}-5x^{\frac{8}{3}}+1184=0.$ *Ans.* $x=8$

48. Find the value of x in the equation

$$(x+12)^{\frac{1}{2}}+(x+12)^{\frac{1}{4}}=6.$$

In this equation we must regard the binomial $(x+12)$ as the unknown quantity; and to simplify the operation we may represent this binomial by y.

$$\text{Then} \quad y^{\frac{1}{2}}+y^{\frac{1}{4}}=6$$

$$\text{which gives } y^{\frac{1}{4}}=2 \text{ or } -3.$$

By restoring the binomial value of y, we have

$$(x+12)^{\frac{1}{4}}=2, \text{ or } -3.$$

Ans. $x=4$.

49. Find the value of x in the equation

$$(2x+6)^{\frac{1}{2}}-6=-(2x+6)^{\frac{1}{4}}.$$

Ans. $x=5$.

50. Find the value of x in the equation

$$x^2+11+\sqrt{x^2+11}=42.$$

Ans. $x=\pm 5$

51. Find the value of x in the equation

$$\frac{1}{(2x-4)^2}=\tfrac{1}{8}+\frac{2}{(2x-4)^4}.$$

Ans. $x=3$, or 1.

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52. Find the value of x in the equation

$$4x^4-4x^3+\frac{x}{2}=33.$$

This equation may be reduced to the form of a *quadratic*, thus.—The first two terms of the *square root* of the first side will be found to be

$2x^2-x$; and the *remainder* will be $-x^2+\frac{x}{2}$, which may be put under the form $-\frac{1}{2}(2x^2-x)$.

Now the *square* of the root found, and the *remainder*, are together equivalent to the first side of the equation; hence we have

$$(2x^2-x)^2-\tfrac{1}{2}(2x^2-x)=33.$$

Ans. $x=2$, or $-1\frac{1}{2}$.

A *Biquadratic* equation may be reduced to the form of a *Quadratic*, as above, whenever the remainder—after having found the first two terms of the square root of the first side—can be resolved into two factors, one of which is the same as *the part of the root* thus found

53. Find the value of x in the equation

$$x^3-8x^2+19x=12.$$

Transposing the 12, and multiplying by x, we have

$$x^4-8x^3+19x^2-12x=0.$$

We have now a Biquadratic equation which may be reduced to the form of a Quadratic by the method just exemplified.

We find the first two terms of the square root of the first side of the Equation, by the common Rule.

$$\begin{array}{r|l}
 & x^4-8x^3+19x^2-12x\ (x^2-4x \\
 & x^4 \\
\hline
2x^2-4x & -8x^3+19x^2 \\
 & -8x^3+16x^2 \\
\hline
 & \qquad\qquad 3x^2-12x.
\end{array}$$

The remainder, $3x^2-12x$, may be resolved into $3(x^2-4x)$, in which the binomial factor is the same as the part of the square root above found.

Then $(x^2-4x)^2+3(x^2-4x)=0$.

Ans. $x=4$, 3, or 1.

54. Find the value of x in the equation

$$x^4-2x^3+x=5112.$$

Ans. $x=9$

55. Find the value of x in the equation

$$x^4+2x^3-7x^2-8x=-12.$$

Ans. $x=2, -3, 1,$ or -2

56. Find the value of x in the equation

$$x^4-10x^3+35x^2-50x+24=0.$$

Ans. $x=1, 2, 3,$ or 4.

57. Find the value of x in the equation

$$x^4-12x^3+44x^2-48x=9009.$$

Ans. $x=13$

58. Find the value of x in the equation

$$x^4-8ax^3+8a^2x^2+32a^3x=s.$$

Ans. $x=2a\pm\sqrt{8a^2\pm\sqrt{s+16a^4}}$.

PROBLEMS

In Pure Equations and Affected Quadratics containing but One Unknown Quantity.

1. Find two numbers such that their product shall be 750, and the quotient of the greater divided by the less, $3\frac{1}{3}$.

Let x represent the greater of the two numbers;
then will $\frac{750}{x}$ represent the less; and the Equation will be

$$x \div \frac{750}{x} \text{ or } \frac{x^2}{750} = 3\tfrac{1}{3}.$$

From this equation we shall find $x=50$, or -50. Each of these values will satisfy the Equation of the problem; but only the *positive* one can be taken to answer the conditions of the problem itself, in which the required numbers are understood to be *positive*, as in the problems of common Arithmetic. *Ans.* 50, and 15.

2. Find a number such that if $\frac{1}{7}$ and $\frac{1}{8}$ of it be multiplied together, and the product divided by 3, the quotient will be $298\frac{2}{3}$. *Ans.* 224.

3. A mercer bought a piece of silk for £16 4*s.*; and the number of shillings that he paid per yard, was to the number of yards, as 4 to 9. How many yards did he buy? and what was the price per yard?

Let x represent the number of shillings he paid per yard;

then $4 : 9 :: x : \frac{9x}{4}$, the number of yards.

But without forming a Proportion, the number of yards is readily known to be $\frac{9}{4}$ of the price per yard.

Ans. 27 yards, at 12*s.* per yard.

4. Find two numbers which shall be to each other as 2 to 3, and the sum of whose squares shall be 208. *Ans.* 8 and 12.

5. A person bought a quantity of cloth for $120; and if he had bought 6 yards more for the same sum, the price per yard would have been $1 less. What was the number of yards? and the price per yard? *Ans.* 24 yards, at $5 per yard.

6. Divide the number 20 into two such parts that the squares of these parts may be in the proportion of 4 to 9. *Ans.* 8, and 12.

7. A merchant bought a quantity of flour for \$100, which he sold again at \$5¼ per barrel, and in so doing gained as much as each barrel cost him. What was the number of barrels? *Ans.* 20.

8. Divide the number 800 into two such parts that the less divided by the greater, may be to the greater divided by the less, as 9 to 25.

Let x represent the less number:—we shall then have the Proportion

$$\frac{x}{800-x} : \frac{800-x}{x} :: 9 : 25;$$

which will be converted into an Equation by putting the product of the two *extremes* equal to the product of the two *means*.

Ans. 300, and 500

9. Two fields which differ in quantity by 10 acres, were each sold for \$2800, and one of them was valued at \$5 an acre more than the other. What was the number of acres in each? *Ans.* 70, and 80.

10. A and B started together on a journey of 150 miles. A traveled 3 miles an hour more than B, and completed the journey 8⅓ hours before him. At what rate did each travel per hour?

Ans. 9, and 6 miles.

11. A man traveled 105 miles, and then found that if he had gone 2 miles less per hour, he would have been 6 hours longer on his journey. At what rate did he travel per hour? *Ans.* 7 miles.

12. A person has two pieces of silk which together contain 14 yards. Each piece is worth as many shillings per yard as there are yards in the piece, and their whole values are in the proportion of 9 to 16; how many yards are there in each piece? *Ans.* 6, and 8 yards.

13. A merchant sold a piece of linen for \$39, and in so doing gained as much per cent. as it cost him. What was the cost of the linen? *Ans.* \$30.

14. A grazier bought as many sheep as cost him \$100. After reserving 5 of the number, he sold the remainder for \$135, and gained \$1 a head on them: how many sheep did he buy? *Ans.* 50.

15. Find two numbers which shall be in the proportion of 7 to 9, and have the difference of their squares equal to 128.

Ans. 14, and 18.

16. An officer would arrange 2400 men in a solid body, so that each rank may exceed each file by 43 men. How many must be placed in rank and file? *Ans.* 75, and 32.

17. Two partners gained £18 by trade. A's money was employed in the business 12 months; and B's, which was £30, 16 months. A received for his capital and gain £26; what was the amount of his capital?

Let x represent A's capital; then $26-x$ will be his gain; and since the gain is in the *compound ratio* of the *capital* and the *time* it was employed, we have

$$12x+16\times 30 : 12x :: 18 : 26-x.$$

The first ratio in this proportion may be simplified by dividing the antecedent and the consequent by 12, (158). *Ans.* £20.

18. A detachment from an army was marching in regular column, with 5 men more in depth than in front; but upon the enemy's coming in sight, the front was increased by 845 men; and by this movement the detachment was drawn up in five lines. What was the number of men? *Ans.* 4550.

19. A company at a tavern had £8 15*s*. to pay, but before their bill was settled, two of them went away, when those who remained had 10*s*. apiece more to pay than before. How many were there in the company at first? *Ans.* 7.

20. Some gentlemen made an excursion, and each one took the same sum. Each gentleman had as many servants as there were gentlemen, and the number of dollars which each had was double the whole number of servants; also the whole sum taken with them was $3456. What was the number of gentlemen? *Ans.* 12.

21. Divide the number 20 into two such parts, that the product of the whole number and one of the parts shall be equal to the square of the other. *Ans.* $10\sqrt{5}-10$, and $30-10\sqrt{5}$.

22. A laborer dug two trenches, one of which was 6 yards longer than the other, for £17 16*s*., and the digging of each cost as many shillings per yard as there were yards in its length. What was the length of each? *Ans.* 10, and 16 yards.

23. There are two numbers whose product is 120, and if 2 be added to the less, and 3 subtracted from the greater, the product of the sum and the remainder will also be 120. What are the two numbers? *Ans.* 8 and 15.

24. Two persons lay out some money on speculation. A disposes of his bargain for £11, and gains as much per cent. as B lays out; B's gain is £36, and it appears that A gains 4 times as much per cent. as B. What sum did each lay out? *Ans.* A £5, B £120.

25. A set out from C towards D, and traveled 7 miles an hour. After he had gone 32 miles, B set out from D towards C, and went each hour $\frac{1}{19}$ of the whole distance; and after he had traveled as many hours as he went miles in one hour, he met A. Required the distance between the two places. *Ans.* 152, or 76 miles.

SOLUTION OF TWO EQUATIONS—ONE OR BOTH OF THE SECOND OR A HIGHER DEGREE—CONTAINING TWO UNKNOWN QUANTITIES.

(264.) For the solution of two Equations, containing two unknown quantities, the method which naturally occurs is,

1. By elimination between the given equations to derive a *new equation* containing but one of the unknown quantities, and thence to find the *value* of that quantity.

2. By substituting this quantity for its symbol in one of the equations containing the other unknown quantity, to determine thence the value of that quantity.

There are, however, some facilitating expedients to be applied, in certain cases, to equations of the second and higher degrees : these will be exemplified as we proceed.

But the solution of two Equations—one or both of the second or a higher degree—containing two unknown quantities, may be impossible by the *method of quadratics*,—from the impossibility of deriving from them a new equation containing but one unknown quantity, which will admit of a quadratic solution

EXAMPLES AND EXERCISES.

1. Find the values of x and y in the equations

$$2x+y=10, \text{ and } 2x^2-xy+3y^2=54.$$

From the first equation, we have

$$x=\frac{10-y}{2}$$

Then $2x^2=\frac{2(100-20y+y^2),}{4}$ and $xy=\frac{10y-y^2}{2}$.

By substituting these values in the second eqnation, we find

$$\frac{2(100-20y+y^2),}{4}-\frac{10y-y^2}{2}+3y^2=54.$$

The value of y may be found from this equation ; and by substituting the value of y for y in the first equation, the value of x may readily be determined.

Observe that the left hand fraction in the last equation may be reduced to *lower terms ;* and the solution of the equation be thus somewhat simplified.

Ans. $x=3$, or $\frac{41}{8}$; $y=4$, or $-\frac{1}{4}$.

2. Find the values of x and y in the equations

$$x+y=9, \text{ and } x^2+y^2=45.$$

Ans. $x=3$, or 6; $y=6$, or 3

☞ Whenever x and y may be interchanged with each other, without changing the *form* of the given equations—as in the preceding example—the two values of one of these letters may be taken, in reverse order, for the two values of the other.

3. Find the values of x and y in the equations

$$xy=28, \text{ and } x^2+y^2=65.$$

Ans. $x=\pm 7$, or ± 4; $y=\pm 4$, or ± 7.

4. Find the values of x and y in the equations

$$x+4y=14, \text{ and } 4x-2y+y^2=11.$$

Ans. $x=2$, or -46; $y=3$, or 15.

5. Find the values of x and y in the equations

$$x+2y=7, \text{ and } x^2+3xy-y^2=23.$$

Ans. $x=3$, or $15\frac{2}{3}$; $y=2$, or $-4\frac{1}{3}$

6. Find the values of x and y in the equations

$$x+\frac{y}{2}=11, \text{ and } xy+2y^2=120.$$

Ans. $x=8$, or $17\frac{2}{3}$; $y=6$, or $-13\frac{1}{3}$.

7. Find the values of x and y in the equations

$$x-y=-2, \text{ and } x+\frac{y}{10}=\frac{3xy}{10}$$

Ans. $x=2$, or $-\frac{1}{3}$, $y=4$, or $1\frac{2}{3}$.

8. Find the values of x and y in the equations

$$x-\frac{x-y}{2}=4, \text{ and } y-\frac{x+3y}{x+2}=1.$$

Ans. $x=2$, or 5; $y=6$, or 3.

9. Find the values of x and y in the equations

$$5(x+y)=13(x-y), \text{ and } x+y^2=25.$$

Ans. $x=9$, or $-14\frac{1}{16}$; $y=4$, or $-6\frac{1}{4}$.

Solutions by Means of an Auxiliary Unknown Quantity.

(265.) When the number of *unknown factors* is the same in every unknown term of the two Equations, the solution will often be facilitated by substituting for one of the unknown quantities the *product of the other into a third unknown quantity.*

10. Find the values of x and y in the equations

$$x^2+xy=54, \text{ and } 2xy+y^2=45.$$

The number of unknown *factors* in each of the unknown terms in these equations, is *two*.

If we assume x to be equal to vy, and substitute this product for x, the given equations will become

$$v^2y^2+vy^2=54,$$

$$\text{and } 2vy^2+y^2=45.$$

From the first of these equations we have

$$y^2=\frac{54}{v^2+v,} \text{ and from the second } y^2=\frac{45}{2v+1}.$$

Putting these two values of y^2 equal to each other,

$$\frac{54}{v^2+v}=\frac{45}{2v+1}.$$

By solving this equation in the usual manner we shall find $v=2$, or $-\frac{3}{5}$; then by substituting these values, successively, in either of the expressions for y^2, we shall find the values of y; and since $x=vy$, we may also readily obtain the value of x.

Ans. $x=\pm6$, or $\mp9\sqrt{-1}$; $y=\pm3$, or $\pm15\sqrt{-1}$.

If in the preceding example an expression for the value of x or y were obtained from either equation, and substituted in the other, the resulting equation, when cleared of radical signs, would be of the *fourth degree;* and we have accordingly found *four values* for each of the unknown quantities, (255.)

11. Find the values of x and y in the equations

$$xy=28, \text{ and } x^2+y^2=65.$$

Ans. $x=\pm7$, or ±4; $y=\pm4$, or ±7.

12. Find the values of x and y in the equations

$$x^2+xy=12, \text{ and } xy-2y^2=1.$$

Ans. $x=\pm3$, or $\pm\frac{8}{\sqrt{6}}$; $y=\pm1$, or $\pm\frac{1}{\sqrt{6}}$

13. Find the values of x and y in the equations

$$4x^2-2xy=12, \text{ and } 2y^2+3xy=8.$$

Ans. $x=\pm 2$, or $\mp 3\sqrt{\frac{1}{7}}$; $y=\pm 1$, or $\pm 8\sqrt{\frac{1}{7}}$.

14. Find the values of x and y in the equations

$$3y^2-x^2=39, \text{ and } x^2+4xy=256-4y^2.$$

Ans. $x=\pm 6$, or ± 102; $y=\pm 5$, or ∓ 59.

Solutions by Means of Two Auxiliary Unknown Quantities.

(266.) When the unknown quantities are *similarly involved* in each of the two Equations, the solution will sometimes be facilitated by substituting for the two unknown quantities the *sum and difference of two other unknown quantities.*

15. Find the values of x and y in the equations

$$x+y=12, \text{ and } \frac{x^2}{y}+\frac{y^2}{x}=18.$$

If we assume x equal to $v+z$, and y equal to $v-z$, we shall have

$$x+y=2v=12; \text{ and hence } v=6.$$

Then $x=6+z$, and $y=6-z$.

Substituting these values of x and y in the second equation, we have

$$\frac{(6+z)^2}{6-z}+\frac{(6-z)^2}{6+z}=18.$$

Clearing this equation of its fractions,

$$(6+z)^3+(6-z)^3=18(36-z^2).$$

By developing both sides of this last equation, and proceeding with the solution in the usual manner, we shall find $z=\pm 2$.

Having now found the values of both v and z, the values of x and y are easily obtained.

Ans. $x=8$ or 4; $y=4$ or 8.

16. Find the values of x and y in the equations

$$x+y=10, \text{ and } x^3+y^3=280.$$

Ans. $x=4$ or 6; $y=6$ or 4.

17. Find the values of x and y in the equations

$$x+y=11, \text{ and } x^4+y^4=2657.$$

Ans. $x=4$ or 7, $y=7$ or 4

18. Find the values of x and y in the equations

$$x+y=10, \text{ and } x^5+y^5=17{,}050.$$

Ans. $x=3$ or 7, $y=7$ or 3

Miscellaneous Solutions and Exercises.

19. Find the values of a and y in the equations

$$xy=6, \text{ and } x^2+x=18-y^2-y.$$

From the second equation,

$$x^2+y^2+x+y=18.$$

Adding *twice* the first,

$$x^2+2xy+y^2+x+y=30.$$

This last equation may be put under the form

$$(x+y)^2+(x+y)=30;$$

which is *quadratic* with reference to $x+y$, and from which we may therefore find the value of $x+y$.

Ans. $x=2$, or 3; or $-3\mp\sqrt{3}$; $y=3$, or 2; or $-3\pm\sqrt{3}$.

20. Find the values of x and y in the equations

$$x+y=6, \text{ and } x^2y^2+4xy=96.$$

Ans. $x=2$, or 4; or $3\mp\sqrt{21}$; $y=4$, or 2; or $3\pm\sqrt{21}$

21. Find the values of x and y in the equations

$$x^{\frac{2}{3}}y^{\frac{3}{2}}=2y^2, \text{ and } 8x^{\frac{1}{3}}-y^{\frac{1}{2}}=14.$$

Dividing the first equation by $y^{\frac{3}{2}}$, we find $x^{\frac{2}{3}}=2y^{\frac{1}{2}}$, or $y^{\frac{1}{2}}=\frac{1}{2}x^{\frac{2}{3}}$, and by substituting this value of $y^{\frac{1}{2}}$ in the second equation, that equation will become *quadratic* with reference to $x^{\frac{1}{3}}$.

Ans. $x=2744$, or 8; $y=9604$, or 4.

22. Find the values of x and y in the equations

$$x^2y-xy=6, \text{ and } x^3y-y=21.$$

Dividing the first equation by the second, we have

$$\frac{x^2y-xy}{x^3y-y}=\frac{6}{21}.$$

This equation will be simplified by reducing each of its two fractional members to its *lowest terms*.

Ans. $x=2$, or $\frac{1}{2}$; $y=3$, or -24.

23. Find the values of x and y in the equations

$$x^3+xy^2=39, \text{ and } x^2y+y^3=26.$$

Ans. $x=3$; $y=2$.

24. Find the values of x and y in the equations

$$x-y=4, \text{ and } x^3-y^3=316.$$

Dividing each side of the second equation by the corresponding side of the first, we shall find, $x^2+xy+y^2=79$.

Squaring the first equation, and subtracting, we have $3xy=63$

Ans $x=7$, or -3; $y=3$, or -7

25. Find the values of x and y in the equations

$$x^2-y^2=5, \text{ and } x^4-y^4=65.$$

Ans. $x=\pm 3$, $y=\pm 2$

26. Find the values of x and y in the equations

$$x+y=60, \text{ and } 2(x^2+y^2)=5xy.$$

The second equation may be put under the form $x^2+y^2-2\frac{1}{2}xy=0$; and the solution will be facilitated by subtracting this from the square of the first equation. *Ans.* $x=40$, or 20; $y=20$, or 40.

27. Find the values of x and y in the equations

$$xy=8, \text{ and } \frac{x^2}{y}+\frac{y^2}{x}=9.$$

Multiplying the two equations together, we find $x^3+y^3=72$; and, multiplying this equation by x^3, we have $x^6+x^3y^3=72x^3$.

From the first equation, $x^3y^3=8^3=512$; and if this number be substituted in the preceding equation we shall have a *quadratic* with reference to x^3. *Ans.* $x=4$, or 2; $y=2$, or 4.

28. Find the values of x and y in the equations

$$xy=25, \text{ and } x^3+y^3=10xy.$$

The second equation will be reduced to the same form as the second in the preceding example, by dividing it by xy.

Ans. $x=5$. $y=5$.

29. Find the values of x and y in the equations

$$x^2-y^2-(x+y)=8, \text{ and } (x-y)^2(x+y)=32.$$

Dividing each equation by $x+y$, we have

$$x-y-1=\frac{8}{x+y}, \text{ and } (x-y)^2=\frac{32}{x+y}.$$

Transposing -1, and squaring, we obtain

$$(x-y)^2=\left(\frac{8}{x+y}+1\right)^2=\frac{32}{x+y};$$

from which will result a *quadratic* with reference to $x+y$.

Ans. $x=5$; $y=3$.

30. Find the values of x and y in the equations

$$x+\sqrt{xy}+y=7, \text{ and } x^2+xy+y^2=21.$$

Dividing the second equation by the first, we have

$$x-\sqrt{xy}+y=3.$$

We shall now obtain two equations of simpler forms by adding the third equation to the first, and subtracting it from the first.

Ans. $x=1$, or 4 ; $y=4$, or 1

31. Find the values of x and y in the equations

$$\frac{x+y}{2}=\sqrt{xy}+4, \text{ and } \sqrt{xy}=\frac{2xy}{x+y}+\frac{12}{5}.$$

From the first equation, by transposition,

$$\sqrt{xy}=\frac{x+y}{2}-4.$$

Clearing this equation of its fraction, and squaring,

$$4xy=(x+y)^2-16(x+y)+64.$$

By equating the two values of $\sqrt{xy}$, from the second and third equations,

$$\frac{2xy}{x+y}+\frac{12}{5}=\frac{x+y}{2}-4;$$

$$\text{or } \frac{2xy}{x+y}=\frac{x+y}{2}-\frac{32}{5}.$$

Clearing this equation of its fractions,

$$4xy=(x+y)^2-\frac{64}{5}(x+y).$$

We shall now obtain a simple equation by subtracting this last equation from the fourth equation.

Ans. $x=2$, or 18; $y=18$, or 2.

32. Find the values of x and y from the equation and *proportion*

$$xy^2-x=3;$$

$$x^2y^4-x^2 : x^2+x^2y^2+x^2y^4 :: 5 : 7.$$

In any proportion the difference of the first and second terms *is to the first*, as the difference of the third and fourth *is to the third*, (160).

Hence from the given proportion, we shall find

$$2x^2+x^2y^2 : x^2y^4-x^2 :: 2 : 5.$$

Dividing the first antecedent and consequent by x^2, (158)

$$2+y^2 : y^4-1 :: 2 : 5.$$

Ans. $x=1$, or $-\frac{6}{5}$; $y=2$, or $\sqrt{-\frac{3}{2}}$.

33. Find the values of x and y in the equations

$$x^2+xy+y^2=13, \text{ and } x^4+x^2y^2+y^4=91.$$

Dividing the second equation by the first,

$$x^2-xy+y^2=7.$$

Adding the third equation to the first, and dividing the result by 2: and also subtracting the third from the first,

$$x^2+y^2=10;$$

$$\text{and} \qquad 2xy=6.$$

The solution now proceeds by adding the latter of these two equations to the former, and also subtracting the latter from the former, and extracting the *square roots* of the resulting equations.

Ans. $x=\pm 3$; $y=\pm 1$.

34. Find the values of x and y in the equations

$$x^2y=x^2y^2-x^2, \text{ and } x^2y^2+x^2=x^3y^3-x^3$$

Dividing each equation by x^2, we have

$$y=y^2-1, \text{ and } y^2+1=xy^3-x.$$

The value of y is to be found from the first of these equations, and substituted in the second. *Ans.* $x=\frac{1}{2}\sqrt{5}$; $y=\frac{1}{2}\pm\frac{1}{2}\sqrt{5}$.

35. Find the values of x and y in the equations

$$xy=x^2-y^2, \text{ and } x^2+y^2=x^3-y^3.$$

If we assume y to be equal to xv, and substitute this product for y in the two equations, (264,) we shall have

$$x^2v=x^2-x^2v^2, \text{ and } x^2+x^2v^2=x^3-x^3v^3.$$

These equations may be solved in the same manner as those in the preceding example. Or the value of x may be found, in terms of y, from the first of the given equations, and substituted in the second.

$$\text{Thus } x=\frac{y}{2}\pm\sqrt{\frac{5y^2}{4}}=\frac{y}{2}\pm\frac{y}{2}\sqrt{5}.$$

Then for the second equation,

$$\left(\frac{y}{2}\pm\frac{y}{2}\sqrt{5}\right)^2+y^2=\left(\frac{y}{2}\pm\frac{y}{2}\sqrt{5}\right)^3-y^3.$$

Ans. $x=\frac{1}{4}(5\pm\sqrt{5})$; $y=\pm\frac{1}{2}\sqrt{5}$.

36. Find the values of x and y in the equations

$$x^3-x^2y-xy^2+y^3=576, \text{ and } x^3+x^2y+xy^2+y^3=2336$$

Subtracting the first equation from the second,

$$2x^2y+2xy^2=1760.$$

Adding this to the second equation, we obtain

$$x^3+3x^2y+3xy^2+y^3=4096.$$

By extracting the *cube root* of this equation, we shall find the value of $x+y$, which may be substituted for $x+y$ in the third equation.

Ans. $x=11$; $y=5$

37. Find the values of x and y in the equations

$$xy=320, \text{ and } x^3-y^3=61(x-y)^3.$$

Dividing the second equation by $x-y$, we have

$$x^2+xy+y^2=61(x-y)^2.$$

By converting this equation into a *proportion*, (153,)

$$x^2+xy+y^2 : (x-y)^2 :: 61 : 1.$$

The solution now proceeds by developing the term $(x-y)^2$,—subtracting each consequent from its antecedent, and forming a proportion of the *consequents* and *remainders*, &c, (160).

Ans. $x=20$; $y=16$.

38. Find the values of x, y, and z, in the equations,

$$x^2+y^2+xy=37;\ x^2+z^2+xz=49;\ y^2+z^2+yz=61.$$

Subtracting the first equation from the second, and *decomposing*,

$$(z-y)(z+y)+(z-y)x, \text{ or } (z+y+x)(z-y)=12;$$

$$\text{from which } z+y+x=\frac{12}{z-y}.$$

Proceeding in like manner with the second and third equations,

$$\text{we shall find } y+x+z=\frac{12}{y-x}.$$

Hence the right hand members of the last two equations are equal to each other, (113.....1); and since the numerators are the same, we have

$$z-y=y-x, \text{ from which } 2y=x+z.$$

By substituting $2y$ for $x+z$ in the sixth equation, we shall find

$$y^2-yx=4.$$

The value of x from this equation, is to be substituted in the first equation. *Ans.* $x=3$; $y=4$; $z=5$.

PROBLEMS

In Quadratic Equations of One or More Unknown Quantities.

1. The *area* of a rectangular lot of ground is 384 square rods, and its length is to its breadth as 3 is to 2. Required the length and breadth of the lot.

The *area*, in square measure, of a rectangle, is expressed by the product of the number of *linear* units in its *length* × the number of linear units in its *breadth.*

The length and breadth must be taken in the same denomination in multiplying: the *area* will be found in the corresponding denomination of *square measure.*

Let x represent the *length,* and y the *breadth* of the lot; then by the conditions of the problem,

$$xy=384,$$

$$\text{and } x : y :: 3 : 2.$$

Or, if x represent the length, $\frac{2x}{3}$ will represent the breadth, and we shall then have

$$\frac{2x^2}{3}=384.$$

Ans. 24, and 16 rods.

2. The length of a rectangular garden exceeds its breadth by 6 rods, and its area is 216 square rods. What are the length and breadth of the garden? *Ans.* 18, and 12 rods.

3. Find two numbers whose sum shall be 24, and whose product shall be equal to 35 times their difference. *Ans.* 14 and 10.

4. Divide a line 20 inches in length into two such parts that the rectangle or product of the whole line and one of the parts shall be equal to the square of the other part.

Ans. $10\sqrt{5}-10$, and $30-10\sqrt{5}$.

5. Find the dimensions of a rectangular field, so that its length shall be equal to twice its breadth, and its area 800 square rods.

Ans. 40, and 20 rods.

6. The sum of the two digits of a certain number is 10, and if their product be increased by 40, the digits will be reversed. What is the number? *Ans.* 46.

7. The sum of two fractions is $1\frac{1}{6}$, and the sum of their *reciprocals* is $3\frac{1}{2}$; what are the two fractions? *Ans.* $\frac{1}{2}$ and $\frac{2}{3}$.

8. The *perimeter*, or sum of the four sides of a rectangle, is 112 rods, and its area is 720 square rods. What are the length and breadth of the rectangle? *Ans.* 36, and 20 rods.

9. Divide the number 60 into two such parts that their product shall be to the sum of their squares as 2 to 5. *Ans.* 20 and 40.

10. A merchant bought a piece of cloth for \$120, and after cutting off 4 yards, sold the remainder for what the whole cost him—by which he made \$1 a yard on what he sold. How many yards did the piece contain? *Ans.* 24.

11. Divide the number 100 into two such parts that the difference of their square roots shall be 2. *Ans.* 64, and 36.

12. A garden which is 20 rods square is surrounded by a walk whose area is equal to $\frac{1}{4}$ of the area of the garden itself. What is the breadth of the walk? *Ans.* $5\sqrt{5}-10$ rods.

13. The sum of the squares of two numbers is 325, and the difference of their squares is 125. What are the numbers?
Ans. 15 and 10.

14. The area of a rectangular court-yard is 875 square rods, and if its length and breadth were each increased by 5 rods, its area would then be 1200 square rods. What are the dimensions of the yard?
Ans. 35 and 25 rods.

15. The difference of two numbers is 4, and the difference of their cubes is 448. What are the two numbers? *Ans.* 8 and 4.

16. A grocer sold 80 pounds of mace and 100 pounds of cloves for £65, and finds that he has sold 60 more of cloves for £20 than of mace for £10. What was the price of each per pound.
Ans. 10*s.* and 5*s.*

17. The fore-wheel of a carriage makes 6 revolutions more than the hind wheel in going 120 yards; but if the circumference of each be increased 1 yard, it will make only four revolutions more in going the same distance. What is the circumference of each wheel?
Ans. 4, and 5 yards.

18. Find four numbers in arithmetical progression, such, that the product of the two *extremes* shall be 45, and the product of the two *means* 77.

Let x be the first term, and y the common difference of the terms; then the numbers will be

$$x,\ x+y,\ x+2y,\ x+3y; \quad (175).$$

and by the conditions of the problem we shall have

$$x^2+3xy=45; \text{ and } x^2+3xy+2y^2=77.$$

Ans. 3, 7, 11, and 15.

19. A farmer has a field 16 rods long and 12 rods wide, which he wishes to enlarge so that it may contain just twice as much area, without altering the proportion of the sides. What will be the dimensions of the field when thus enlarged? *Ans.* $16\sqrt{2}$; and $12\sqrt{2}$.

20. Find three numbers, such, that the difference of the first and second shall be two less than the difference of the second and third, their sum 17, and the sum of their squares 115.

The solution will be facilitated by assuming x to represent the second number, and y the difference of the first and second.
Ans. 3, 5, and 9.

21. There are two square gardens which together contain 1025 square rods, and a side of the one exceeds a side of the other by 5 rods. What are the sides of the two gardens? *Ans.* 20, and 25 rods.

22. Find two numbers, such, that their sum, their product, and the difference of their squares shall all be equal to one another.

Take $x+y$ to represent the greater, and $x-y$ the less number.
Ans. $\frac{3}{2}\pm\sqrt{\frac{5}{4}}$, and $\frac{1}{2}\pm\sqrt{\frac{5}{4}}$.

23. A merchant received $12 for a quantity of linen, and an equal sum, at 50 cents less per yard, for a quantity of calico, which exceeded the quantity of linen by 32 yards. What was the quantity of each?
Ans. 16, and 48 yards.

24. Find two numbers whose sum multiplied by the greater shall be equal to 192, and whose difference multiplied by the less shall be equal to 32.

The solution will be facilitated by taking x to represent one of the required numbers, and xy the other. *Ans.* 12, and 4.

25. Three merchants gained $1444; of which their respective shares were such that B's, added to the square root of A's, made $920: but if added to the square root of C's it made $912. What was the share of each? *Ans.* $400, $900, and $144.

26. The sum of three numbers in harmonical progression is 13, and the product of the two extremes is 18. What are the numbers?

If x and y represent the two extremes, the mean term will be

$$\frac{2xy}{x+y} \quad (184).$$

Ans. 6, 4 and 3.

27. There is a rectangular field whose length is to its breadth as 4 to 3. A part of this field, which is equal to $\frac{1}{4}$ of the whole, being in meadow, there remain for ploughing 1296 square rods. What are the dimensions of the field? *Ans.* 48, and 36 rods.

28. The sum of three numbers in geometrical progression is 21, and the sum of their squares is 189. What are the numbers?

If x and y represent the two extremes, the mean term will be

$\sqrt{xy}$, (189). *Ans.* 3, 6, and 12.

29. A and B set out from two places which are distant 110 miles, and traveled towards each other. A went five miles an hour; and the number of hours in which they met was greater, by four, than the number of miles B went per hour. What was B's rate of traveling?

Ans. 6 miles per hour.

30. The arithmetical mean between two numbers exceeds the geometrical mean by 13, and the geometrical mean exceeds the harmonical mean by 12. What are the numbers? *Ans.* 234 and 104.

31. Three merchants made a joint stock, by which they gained a sum less than that stock by \$80. A's share of the gain was \$60, and his contribution to the stock was \$17 more than B's; also B and C together contributed \$325. How much did each contribute?

Ans. \$75, \$58, and \$267

32. Of three numbers in geometrical progression the greatest exceeds the least by 15, and the difference of the squares of the greatest and the least is to the sum of the squares of the three numbers as 5 to 7. What are the numbers?

Assume x to represent the first term, ond y the *ratio* of the progression. *Ans.* 5, 10, and 20.

33. Two persons set out from different places, and traveled towards each other. On meeting, it appeared that A had traveled 24 miles more than B, and that A could have gone B's journey in 8 days, while B would have been 18 days in performing A's journey. What distance was traveled by each? *Ans.* 72, and 48 miles.

34. The joint stock of two partners was \$416. A's money was in the business 9 months, and B's 6 months. When they shared stock and gain, the first received \$228, and the second \$252; what was each man's amount of stock? *Ans.* A's \$192, B's \$224.

35. The sum of \$700 was divided among four persons, A, B, C, and D, whose shares were in geometrical progression; and the difference between the greatest and the least was to the difference between the two means as 37 to 12. What were the several shares?

Ans. \$108, \$144, \$192, and \$256.

SOLUTION OF AFFECTED CUBIC AND HIGHER EQUATIONS.

Various methods have been devised for the solution of Affected Equations of the third and higher degrees. Some of these methods are very prolix,—while others are of limited application; we shall explain those which are the most useful in a practical point of view, without attempting a full exposition of this subject.

Under the head of General Properties of Equations, we have already noticed the *divisors*, (253), and the number of *roots*, (255) of equations; we here present the

General Law of the Coefficients of Equations.

(267) When the terms of any *complete* Equation containing but one unknown quantity x, are all arranged, according to the descending powers of x, in the first member—with the known or absolute term for the *last term*—and the coefficient of the first term is *unity;* then

1. The coefficient of the second term is equal to the sum of all the *roots* of the equation, with their *signs* changed.

2. The co-efficient of the third term is equal to the sum of the *products* of all the roots *combined two* and *two*, with their signs changed, &c.

3. The known or absolute term is equal to the product of all the roots, with their signs changed.

To demonstrate these principles with reference to a Cubic Equation, let the three roots be denoted by a, b, and $-c$; then $x-a$, $x-b$, and $x+c$ are the *divisors* of the equation, and the equation may be accordingly resolved into

$$(x-a)\ (x-b)\ (x+c)=0,\quad (253).$$

By performing the multiplication which is here indicated, and decomposing the terms containing the like powers of x in the product, we find

$$x^3+(c-a-b)x^2+(ab-ac-bc)x+abc=0.$$

In this equation the coefficient of x^2 is the sum of the roots a, b, and $-c$, with their signs changed; the coefficient of x is the sum of the products of the roots combined *two* and *two*, with their signs changed; and the known or absolute term abc is the product of all the roots, with their signs changed.

The same principles may be demonstrated, in like manner, in reference to an Equation of the second, or of any of the higher degrees.

An application of these principles may be made to the equation $x^2+3x-10=(x-2)(x+5)=0$, whose roots are 2, and -5; or to $x^3-19x+30=(x-2)(x+5)(x-3)=0$, whose roots are 2, -5, & 3

In the second equation it will be observed that the second term containing x^2, is wanting, since its coefficient, that is, the sum of the roots 2, -5, and 3, is 0; and that $19x$ therefore corresponds to the third term, (267...2).

Determination of the Integral Roots of Equations.

(268.) If an equation containing but one unknown quantity x, with all its terms transposed to one side, be divisible by $x+$ or $-$any *number*, that number, with a contrary sign, will be a *root* of the equation (254).

The *trial numbers* to be used in this division, are the factors or *divisors* of the known term of the equation, since that term is equal to the product of all the roots of the equation, (267...3).

When an equation has any *integral roots*, such roots may be readily determined by an application of these principles.

EXAMPLE.

To find the values of x in the equation

$$x^3+3x^2-4x=12.$$

The divisors of the known term 12 are 1, 2, 3, 4, 6, and 12; and it will be found, on trial, that the equation

$$x^3+3x^2-4x-12=0,$$

is divisible by $x-2$, $x+2$, and $x+3$; hence the values of x, or roots of the equation, are 2, -2, and -3.

After any one of the three roots has been determined, the two remaining ones may be obtained directly from the *quadratic* equation which results from dividing the given equation by $x+$ or $-$ the root already found.

$$x-2)x^3+3x^2-4x-12(x^2+5x+6.$$

By dividing the given equation by $x-2$, we thus find

$$x^2+5x+6=0,$$

$$\text{or } x^2+5x=-6, \text{ which gives } x=-2 \text{ or } -3$$

By this method the last two roots are found the same as before.

SOLUTION OF EQUATIONS BY APPROXIMATION.

(269.) The following method of solution may be applied to an Equation of any degree—even to one in which the unknown quantity is left, without rationalization, in a *surd* expression.

1. By trial find two numbers—differing by a unit or less—which being substituted for x in the given Equation, will produce results, the one less and the other greater than the *known term* of the equation; then,

The difference between the *two results*,
Is to the difference between the *two assumed numbers*,
As the difference between *either result* and the *known term*,
Is to the *correction*, nearly, required in the corresponding assumed number.

2. Take the corrected root thus obtained for one of two numbers to be substituted for x, and find, and apply, a *correction* as before.

We shall thus obtain a *nearer* value of the unknown quantity; and the approximation may be carried, in like manner, to any required exactness.

EXAMPLE.

To find an approximate value of x in the equation.

$$x^3+x^2+x=100.$$

First, It will be found that x is more than 4, and less than 5. Substituting these numbers for x, we have

64.......x^3.......125
16.......x^2....... 25
4.......x 5

84 155

The difference between the two results is $155-84=71$; and the difference between the *less result* and the known term 100 is 16.

Then 71 : 1 : : 16 : the *correction* . 225.

This correction, added to the *less* assumed number, gives 4.225 for an *approximate* value of x.

Secondly, By substituting 4.2 and 4.3 for x, we have

74.088......x^3......79.507
17.64 ......x^2......18.49
4.2 ......x 4.3

95.928 102.297

Forming a proportion between the difference of these **two results**, and the difference between the *greater result* and the known term 100,

6.369 : ,1 : : 2.297 : the *correction* .036.

By subtracting this correction from the *greater* assumed number 4 3, we have 4.264 for a *nearer* value of x.

For the next approximation we should take 4.264 and 4,265 to to be substituted for x in the given Equation. The value of x would then be found to be 4.2644299 *very nearly.*

In the Proportion for finding the *correction*, it is best to employ the *less error* in the results of the substitution.

Thus in the first substitution, in this Example, the error in the less result 84 is $(100-84)=16$, and this being less than the error in the 155, we employ 16 in the first proportion.

But in the second substitution, the error in the greater result 102.297 is less than the error in the 95.928, and we accordingly use $(102.297-100)=2.297$ in the second proportion.

Each approximative solution will generally *double the number of true figures in the root.* Thus in the preceding Example we found by trial that 4 is the first figure in the root, and the first solution gives 4.2 for the first two correct figures; the next solution gives 4.264; and the number of figures will again be doubled by a third solution. This property determines the number of figures which need be found in the successive *corrections* of the assumed numbers.

To find the other Roots of the given Equation, we would divide

$$x^3+x^2+x-100=0 \text{ by } x-4{,}2644, \text{ \&c.}=0, \quad (253).$$

We should thus obtain a *quadratic* equation, from which the other two values of x might be determined, according to the usual method.

☞ When all the Roots of an Equation have been found, we may *verify* them by the property that, with their signs changed, their sum must be equal to the *coefficient of the second term of the equation*, (267...1).

Thus the sum of the *three roots* of the equation in the preceding Example, with their signs changed, would be *unity*, which is the coefficient of the second term x^2.

EXERCISES

On Affected Cubic and Biquadratic Equations.

1. Find the values of x in the equation

$$x^3-6x^2+11x=6, \quad (268).$$

Ans. $x=1$, 2, or 3.

2. Find the values of x in the equation

$$x^3-9x^2+26x=24.$$

Ans. $x=2$, 3, or 4.

3. Find the values of x in the equation

$$x^3-3x^2-6x=-8.$$

Ans. $x=1$, 4, or -2.

4 Find the values of x in the equation

$$2x^3-6x^2-8x=-24.$$

Ans. $x=2$, 3, or -2.

5. Find the values of x in the equation

$$x^4+2x^3-13x^2-14x+24=0.$$

Ans. 1, -2, 3, or -4

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6. Find an approximate value of x in the equation

$$x^3+10x^2+5x=260, \quad (269).$$

Ans. $x=4.117'$.

7. Find an approximate value of x in the equation

$$x^3-15x^2+63x=50.$$

Ans. $x=1.028.'$

8. Find an approximate value of x in the equation

$$x^3-17x^2+54x=350.$$

Ans. $x=14.95'$.

9. Find an approximate value of x in the equation

$$x^4-3x^2-75x=10000.$$

Ans. $x=10.23'$.

10. Find an approximate value of x in the equation

$$2x^4-16x^3+40x^2-30x+1=0.$$

Ans. $x=1.284'$.

11. Find an approximate value of x in the equation

$$(\tfrac{1}{5}x^2-15)^2+x\sqrt{x}=90.$$

Ans. $x=10.52'$.

ELIMINATION BY THE METHOD OF COMMON DIVISOR.

(269.) The following is a general method of Elimination, and, for Equations of the *higher degrees*, it will sometimes be found preferable to any other.

Transpose all the terms of the two Equations to one side; then divide one into the other, and the *remainder* into the *divisor*, and so on, as in finding the Greatest Common Measure, (66,) until one of the two unknown quantities is *eliminated from the remainder*; and put this remainder $=0$.

To eliminate x from the equations

$$x^2+xy=10, \text{ and } xy+2y^2=24.$$

$$\begin{array}{r|l}
 & xy+2y^2-24 \\
 & x \\
\cline{2-2}
x^2+xy-10 & x^2y+2xy^2 24-x\ (y \\
 & x^2y+xy^2-10y \\
\cline{2-2}
 & \qquad xy^2-24x+10y, \text{ or } x(y^2-24)+10y.
\end{array}$$

$$\begin{array}{r|l}
 & x^2+xy-10 \\
 & y^2-24 \\
\cline{2-2}
x(y^2-24)+10y & x^2(y^2-24)+x(y^3-24y)-10y^2+240\ (x+y \\
 & x^2(y^2-24)+10xy \\
\cline{2-2}
 & \qquad\qquad x(y^3-24y)-10xy-10y^2+240 \\
 & \qquad\qquad x(y^3-24y)+10y^2 \\
\cline{2-2}
 & \qquad\qquad\qquad\qquad -10xy-20y^2+240
\end{array}$$

$$\begin{array}{r|l}
 & x(y^2-24)+10y \\
 & y \\
\cline{2-2}
xy+2y^2-24 & x(y^3-24y)+10y^2\ (y^2-24 \\
 & x(y^3-24y)+2y^4-48y^2-24y^2+576 \\
\cline{2-2}
 & \qquad\qquad -2y^4+82y^2-576=0.
\end{array}$$

In the remainder $-10xy-20y^2+240$, we cancel the factor 10, and change the *signs*, for the next divisor. In dividing into this divisor, we take the binomial y^2-24 for the quotient, and multiply the divisor by this binomial.

The first remainder is equal to 0, because the divisor and dividend are each equal to 0; and it follows hence that *each subsequent remainder is equal to* 0.

The operation will be much more simple if we divide the first equation by the second: the result will be the same.

CHAPTER XI.

GENERAL DESCRIPTION OF PROBLEMS.—INEQUATIONS.

MISCELLANEOUS PROBLEMS.

1. *Determinate Problems.*

(270.) A *Determinate Problem* is one in which the given conditions determine the values of the unknown or required quantities.

A Determinate Problem is represented by as many *independent* equations as there are different conditions to be expressed, or unknown quantities to be determined, (120.)

All the Problems which have hitherto been proposed in this work, are *determinate;* and no example of this kind need be here given.

2. *Indeterminate Problems.*

(271.) An *Indeterminate* Problem is one in which the given conditions do not determine the values of the required quantities,—admitting either of an *unlimited* number of values to those quantities, or else of a variety of values, within certain limits.

An Indeterminate Problem is represented either by a less number of independent Equations than there are unknown quantities to be determined, or by an *identical* equation.

We give an example of each of these forms of indeterminateness.

EXAMPLE I.

To find three numbers such that the first shall be 5 less than the second, and the sum of the second and third shall be 12.

This Problem contains but two conditions; and if we represent the three required numbers by x, y, and z, we shall have only the two Equations

$$y-x=5;$$
$$y+z=12.$$

By subtracting the first equation from the second, we have

$$x+z=7.$$

This equation will admit of an *unlimited* number of values of x and z; for we may assume any value whatever for one of the letters, as x, and determine thence the corresponding value of z.

Thus if $x=\frac{1}{4}$, $z=6\frac{3}{4}$: if $x=\frac{1}{2}$, $z=6\frac{1}{2}$; if $x=\frac{3}{4}$, $z=6\frac{1}{4}$, &c.; and from the values of x or z, we might obtain the corresponding values of y from one of the given equations.

If, however, the required numbers were limited to *integral* values. the third equation would be satisfied only by

$$x=1,\ 2,\ 3,\ 4,\ 5,\text{ or }6,\text{ and }z=6,\ 5,\ 4,\ 3,\ 2,\text{ or }1.$$

In the first equation $y=5+x$, which would give

$$y=6,\ 7,\ 8,\ 9,\ 10,\text{ or }11.$$

EXAMPLE II.

To find a number such, that $\frac{3}{4}$ of it, diminished by $\frac{1}{3}$ of it, and by 5, shall be equal to $\frac{1}{12}$ of the excess of 5 times the number above 60

The equation of this problem will be

$$\frac{3x}{4}-\frac{x}{3}-5=\frac{5x-60}{12}.$$

Clearing the equation of its fractions,

$$9x-4x-60=5x-60;$$

$$\text{or }\ 5x-60=5x-60.$$

This last is an *identical* Equation, which will be satisfied by attributing to x any numerical value whatever. The problem is therefore entirely *indeterminate.*

We may obtain an expression for the value of x from the last equation. Thus, by transposition,

$$5x-5x=60-60.$$

By adding similar terms, and retaining x as a symbol in the first member, we have

$$0x=0;$$

$$\text{which gives }\ x=\tfrac{0}{0}.$$

Hence $\frac{0}{0}$ is a symbol of an *indeterminate* quantity.

The same thing will appear from considering that the *quotient* of $0\div 0$ is *any quantity whatever;* inasmuch as the *divisor* $0\times$ *any quantity* will produce the *dividend* 0, (43).

3. *Impossible Problems.*

(272.) An *Impossible* Problem is one in which there is some condition, expressed or implied, which *cannot be fulfilled.*

An Impossible Problem is represented by a greater number of *independent* equations than there are unknown quantities to be determined; or by an equation in which the value of the unknown quantity is *negative—zero—infinite*—or *imaginary.*

We subjoin an example of each of these forms of impossibility.

EXAMPLE I.

To find two numbers whose sum shall be 10, difference 2, and product 20.

Representing the two numbers by x and y, we shall have

$$x+y=10; \qquad x-y=2; \qquad xy=20.$$

From the first and second equations the values of x and y will be found to be $x=6$, and $y=4$. The third equation cannot, therefore, be fulfilled; that is, the problem is *impossible.*

If the third equation were $xy=24$, the problem would be possible, but this would not be an *independent* equation, since it may be derived from the other two.

Thus, squaring the first and second equations, and subtracting, we find

$$4xy=96, \text{ or } xy=24.$$

EXAMPLE II.

To find a number which, added to 17 and to 53, will make the first sum equal to $\frac{1}{4}$ of the second.

If x represent the number, the equation will be

$$17+x=\frac{53+x}{4}.$$

From this equation we shall find $x=-5$. This number, added to 17 and 53, gives 12 and 48, and $12=\frac{1}{4}$ of 48.

The problem is *impossible* in an arithmetical sense, according to which addition always implies *augmentation;* and it is in this sense only that the problem would be considered.

To make it arithmetically consistent, it should be stated thus:

To find a number which, *subtracted* from 17 and from 53, will make the first *remainder* equal to $\frac{1}{4}$ of the second.

EXAMPLE III.

To find a number such, that if 82 be increased by 3 times that number, $\frac{1}{6}$ of the sum will be equal to $13\frac{2}{3}$.

If x represent the number, the equation will be

$$\frac{82+3x}{6}=13\tfrac{2}{3}.$$

Clearing the equation of its fractions, we find

$$82+3x=82;$$

which gives $3x=82-82=0$;

and $x=\frac{0}{3}=0$, (50).

Hence no number can be found that will fulfil the conditions of the problem; that is, the problem is *impossible*. The result shows that $\frac{1}{6}$ of 82 itself is equal to $13\frac{2}{3}$.

EXAMPLE IV.

To find a number such, that the sum of $\frac{1}{4}$ of it and $\frac{2}{3}$ of it, diminished by 2, shall be equal to $\frac{11}{12}$ of it increased by 3.

If x represent the number, the equation will be

$$\frac{x}{4}+\frac{2x}{3}-2=\frac{11x}{12}+3.$$

From this equation we shall find

$$3x+8x-24=11x+36; \text{ or } 11x-11x=0x=60;$$

and $x=\frac{60}{0}=\infty$, *infinity;* (50).

The result shows that it would require a number *infinitely great*, to fulfil the conditions of the problem. The problem is therefore *impossible.*

EXAMPLE V.

To divide the number 24 into two such parts, that their product shall be 150.

If x represent one of the two parts, $24-x$ will represent the other and the equation will be

$$24x-x^2=150$$

or $x^2-24x=-150$, (117);

which gives $x=12\pm\sqrt{144-150}$,

$$=12\pm\sqrt{-6}.$$

In this value of x, the part $\sqrt{-6}$ is *imaginary*, that is, it is an *impossible* quantity, (246); hence the problem is impossible.

That the preceding problem is impossible, will also appear from the following general proposition; viz:

(273.) The *square of one half* of any quantity is greater than the product of any *two unequal parts* of the quantity.

Let s represent any number, and d the difference between any two parts of the number; then the respective parts are

$$\frac{s+d}{2} \text{ and } \frac{s-d}{2} \text{ (168), and their } \textit{product} \text{ is } \frac{s^2-d^2}{4}.$$

This product will vary directly as its numerator s^2-d^2 (147); and will therefore have its *greatest value* when $d=0$. In that case the product becomes $\frac{1}{4}s^2$, which is the square of $\frac{1}{2}s$, half the given number.

It may also be remarked here, that *the sum of the two equal factors* of any quantity, *is less than the sum of any two unequal factors* into which the quantity can be resolved.

For, s representing the *sum*, and d the difference, of the two factors of a quantity, those factors are

$$\frac{s+d}{2} \text{ and } \frac{s-d}{2} \text{ (168), and their } \textit{product} \text{ is } \frac{s^2-d^2}{4}.$$

This product will retain a *constant value*, if s^2 and d^2 be *equally diminished*, for then the value of s^2-d^2 will remain *constant*. In this diminution, s^2 will have its *least value* when d^2 is made 0; so that s, the sum of the two factors, will have its least value when $a=0$, and the two factors will then be *equal to each other*.

Signification of the Different Forms under which the Value of the Unknown Quantity may be found in an Equation.

(274.) 1. *Positive* values of the unknown or required quantities, fulfil the conditions of problems in the sense in which they are proposed.

2. A value of the unknown quantity of the form $\frac{0}{0}$, shows that the problem from which the equation was derived is *indeterminate*.

3. A *negative* value of the unknown quantity, in an equation of the *first degree*, indicates an *impossibility* in the problem, produced by taking this quantity *additively*, instead of *subtractively*, or *vice versa*.

4. When the value of the unknown quantity in an equation is *zero*, *infinite*, or *imaginary*, the problem from which the equation was derived is *impossible*.

INEQUATIONS.

(275.) An INEQUATION is an expression of the *inequality* between two quantities by means of the sign $>$, *greater than*, or $<$, *less than*, placed between them.

Thus $a>x+y$ denotes that a is greater than $x+y$;
$x-y<b$ denotes that $x-y$ is less than b.

It will be observed that the sign opens towards the *greater* quantity.

(276.) Inequations are employed in the solution of Problems, whose conditions involve *inequalities*, and render the required quantities *indeterminate* within certain necessary limits.

They are also sometimes employed to determine the *sense* in which an inequality between given quantities subsists.

(277) Two Inequalities are said to subsist in the *same sense*, when the greater quantity stands on the right of the sign in both, or on the left in both; otherwise, the Inequalities subsist in a *contrary sense.*

In $a>b$ and $y>x$ the Inequalities subsist in the *same sense,*
in $a>b$ and $x<y$ the Inequalities subsist in a *contrary sense*

Inequalities between Negative Quantities.

(278.) Any disconnected *negative* quantity, as $-a$, may be regarded as $0-a$; and of two such quantities, that one is therefore the *less,* algebraically, which, if both were positive, would be the *greater.*

Thus -5 is, algebraically, less than -3; that is, $0-5$ is less than $0-3$. The consistency of this will appear from considering, that the greater the quantity *subtracted,* the less will be the *remainder.*

A negative quantity being thus regarded as the *remainder* when the corresponding positive quantity is subtracted from 0, is sometimes said to be *less than* 0, and therefore less than any positive quantity. This must be understood, however, as merely implying the *contrariety* of a *negative* to a positive quantity, in its effect upon a calculation; for a quantity, considered in itself, can never be less than nothing.

Transformation of Inequations.

(279.) An Inequation may be transformed by *clearing it of fractions*, by the *transposition and addition of terms*, &c., in the same manner as an Equation; and the result will generally preserve the Inequality in the *same sense*. But,

I. If both sides of an Inequation be multiplied or divided by a *negative* quantity, the resulting Inequality will subsist in a *contrary sense*.

Thus by multiplying both sides of the Inequation $5>3$ by -2, we have the *contrary* Inequality $-10<-6$. (278.)

II. If the signs of all the terms of an Inequation be changed, $+$ to $-$ and $-$ to $+$, the Inequality will be changed to the *contrary sense*.

This follows from the preceding proposition, since the changing of all the signs, $+$ and $-$, is equivalent to multiplying both sides of the Inequation by -1.

III. Like powers or roots of two sides of an Inequation will sometimes form an Inequality subsisting in a *contrary sense*.

Thus by squaring both sides of $-3>-5$ we have $9<25$, in which the Inequality subsists in a contrary sense.

And by extracting the square root of both sides of $25>16$, we have $5>4$, or $-5<-4$, in the latter of which the Inequality is changed to the contrary sense.

IV. If the corresponding sides of two Inequations, subsisting in the same sense, be subtracted the one from the other, the result will sometimes form an Inequality in the *contrary sense*.

Thus by subtracting $10>5$ from $12>9$, we have $2<4$, in which the Inequality subsists in a contrary sense.

An Example of the use of Inequations may be given in showing that the *sum of the squares of any two unequal quantities is greater than twice the product of the quantities.*

Since every even power of a quantity is *positive*, we have

$$(a-b)^2 \text{ or } a^2-2ab+b^2>0.$$

Adding $2ab$ to each side of the Inequation, we find that

$$a^2+b^2>2ab.$$

Exercises on Inequations.

1. Find a number such, that when multiplied by 5, the product shall be less than 40, and when 5 is added to 3 times the square of the number, the sum shall be greater than 80.

$$5x<40, \text{ and } 3x^2+5>80.$$

Proceeding in the same manner as with *Equations*, we shall find x to be <8, and >5.

The required number is therefore *indeterminate within the limits* 8 and 5; that is, it is 7, or 6, or either of these numbers *plus* any proper fraction.

2. What number is that whose third part diminished by 3 is greater than 20, and whose fourth part increased by 4 is less than 30?

Ans. Any number between 69 and 104.

3. Find a number whose square diminished by 10 is less than 90, and whose square root increased by 2 is greater than 5.

Ans. 9 plus any proper fraction.

4. Find a number such, that if it be multiplied by 2, 3, and 4, successively, the sum of the products shall be greater than 100, and if it be divided by the same numbers, the sum of the quotients shall be less than 30.

Ans. Any number between $11\frac{1}{9}$ and $27\frac{9}{13}$

5. A farmer sold a number of cattle. If 6 be subtracted from 3 times the number, the remainder will be less than the number increased by 48; and if 6 be added to 4 times the number, the sum will be greater than 27 increased by 3 times the number; what was the number of cattle?

Ans. 22, 23, 24, 25, or 26.

6. Find a number whose square added to 4 times the number itself shall be more than 45, but whose square diminished by 10 times the number shall be less than 75.

Ans. Any number between 5 and 15.

7. Five times the number of miles between two places increased by 10, is less than 100; and 8 times the number diminished by 5, is greater than the number increased by 100. Required the number of miles between the two places.

Ans. Any number between 15 and 18.

MISCELLANEOUS PROBLEMS.

1. A, B, and C together have $2000. B has $100 less than twice as much as A, and C $400 less than twice as much as the other two together. What sum has each?
Ans. A, $300; B, $500; C, $1200.

2. A gentleman has three plantations. The first contains 250 acres, the second as much as the first and $\frac{1}{6}$ of the third, and the third as much as the first and second. What is the whole number of acres?
Ans. 1200 acres.

3. A company of workmen had been employed on a piece of work for 24 days, and had half finished it, when, by calling in the assistance of 16 more men, the remaining half was completed in 16 days. What was the original number of men? *Ans.* 32 men.

4. A's money was equal to $\frac{3}{4}$ of B's. A paid away $50 less than $\frac{2}{3}$ of his, and B $50 more than $\frac{3}{4}$ of his, when it was found that the latter had remaining only $\frac{1}{3}$ as much as the former. What sum had each at first? *Ans.* A, $300; B, $400.

5. A person wishing to enclose a piece of ground with palisades, found, that if he set them one foot asunder, he would not have enough by 150, but if he set them one yard asunder, he would have too many by 70. What was the number of his palisades?
Ans. 180 palisades.

6. From two tracts of land of equal size, were sold quantities in the proportion of 3 to 5. If 150 acres less had been sold from the one which is now the smaller of the two, only $\frac{2}{3}$ as much would have been taken from it as from the other; how many acres were sold from each? *Ans.* 150, and 250 acres.

7. A and B had adjoining farms, which, in quantity, were in the ratio of 4 to 5. A sold to B 50 acres, and afterwards purchased from B one-third of his entire tract, when it was found that the original ratio of their quantities of land had been *reversed*. How many acres had each at first? *Ans.* A, 200; B, 250 acres.

8. A waterman can row down the middle of the stream, on a certain river, 5 miles in $\frac{3}{4}$ of an hour; but it takes him $1\frac{1}{2}$ hours to return, though he keeps along shore, where the current is but half as strong as in the middle. What is the velocity of the middle of the stream? *Ans.* $2\frac{2}{9}$ miles per hour.

9. A farmer has three flocks of sheep, whose numbers are in the proportion of 2, 3, and 5. If he sell 20 from each flock, the whole number will be diminished in the proportion of 4 to 3; how many has he in each flock? *Ans.* 48, 72, and 120 sheep

10. The sum of $1170 is to be divided between three persons, A, B, and C, in proportion to their ages. Now A's age is to B's as 1 to $1\frac{1}{3}$, and to C's as 1 to 2; what are the respective shares?

Ans. $270; $360; $540.

11. A hare is 50 leaps before a greyhound, and takes 4 leaps to the greyhound's 3; but 2 of the greyhound's leaps are as much as 3 of the hare's. How many leaps must the greyhound take to catch the hare?

Ans. 300.

12. A vintner has two casks of wine, the contents of which are in the proportion of 5 to 6, and if $\frac{1}{6}$ of the quantity in the second were to be drawn off, the contents of the two casks would be equal. How many gallons are there in each?

Ans. This problem is *indeterminate;* how is its indeterminateness indicated?

13. A person looking at his watch, and being asked what o'clock it was, replied that it was between *eight* and *nine*, and that the hour and minute hands were exactly together. What was the time?

Ans. 43*m.* $38\frac{2}{11}$*s.* past *eight.*

14. A criminal having escaped from prison, traveled 10 hours before his escape was known. He was then pursued, and gained upon 3 miles an hour. When his pursuers had been 8 hours on the way they met an express going at the same rate as themselves, who had met the criminal 2 hours and 24 minutes before. In what time from the commencement of the pursuit will the criminal be overtaken?

Ans. 20 hours.

15. A regiment of militia containing 875 men is to be raised from three counties, A, B, and C. The quotas of A and B are in the proportion of 2 and 3, and of B and C in the proportion of 4 to 5. What is the number to be raised by each?

Ans. 200, 300, and 375 men.

16. If 19 pounds of gold, in air, weighs 18 pounds in water; 10 pounds of silver, in air, weighs 9 in water; and a mass of 106 pounds, composed of gold and silver, weighs 99 pounds in water; what are the respective quantities of gold and silver in the mass?

Ans. 76, and 30 pounds.

17. A farmer having mixed a certain quantity of corn and oats, found that if he had taken 6 bushels more of each, there would have been 7 bushels of corn to 6 of oats; but if he had taken 6 bushels less of each, there would have been 6 bushels of corn to 5 of oats. How many bushels of each were mixed?

Ans. 78, and 66 bushels.

18. Two persons, A and B, can perform a piece of work in 16 days. They work together for 4 days, when A being called off, B is left to finish it, which he does in 36 days more. In what time could each do it separately?

Ans. 24, and 48 days.

19. A merchant has two casks containing unequal quantities of wine. Wishing to have the same quantity in each, he pours from the first into the second as much as the second contained at first; then he pours from the second into the first as much as was left in the first; and then again from the first into the second as much as was left in the second, when there are found to be 16 gallons in each cask. How many gallons did each cask contain at first? *Ans.* 22, and 10 gallons.

20. A fisherman being asked how many fish he had caught, replied, If 5 be added to one-third of the number that I caught yesterday, it will make half the number I have caught to-day; or if 5 be subtracted from three times this half, it will leave the number I caught yesterday. How many were caught each day?

Ans. This problem is *impossible;* how is its impossibility indicated?

21. A laborer engaged for n days, on condition that he should receive p pence for each day that he worked, and forfeit q pence for each day that he idled. At the end of the time he received s pence; how many days did he work? and how many was he idle?

Ans. Worked $\frac{nq+s}{p+q}$; was idle $\frac{np-s}{p+q}$ days.

22. A, B, and C engage in a joint speculation. A invests $2000 for 5 months, B $2400 for 4 months, and C $1600 for 7 months, The profits amount to $4620; what is each man's share of profit?

Ans. $1500, $1440, $1680.

23. A farmer wishes to mix rye worth 40 cents a bushel, and oats worth 26⅔ cents a bushel, in such quantities as to produce 100 bushels which shall be worth 30 cents a bushel. What quantity of each must be taken? *Ans.* 25, and 75 bushels.

24. Three persons engage in a joint mercantile adventure, in which the first has the capital a for the time b, the second the capital c for the time d, and the third the capital e for the time f. Their profits amount to s; what is each partner's share of profit?

Ans. $\frac{abs}{ab+cd+ef}$; $\frac{cds}{ab+cd+ef}$; $\frac{efs}{ab+ed+ef}$.

25. A church which cost $40,000 is insured, annually, at 1½ per cent., for such an amount, that, in case of its being destroyed by fire, the Insurance Company shall be liable for the cost of the edifice, and the *premium* of insurance. What is the sum insured?

Ans. 40609.13'.

26. Four towns are situated in the order of the first four letters of the alphabet. The distance from A to D is 34 miles; the distance from A to B is to the distance from C to D as 2 to 3; and ¼ of the distance from A to B added to half the distance from C to D, is 3 times the distance from B to C. What are the respective distances?

Ans. 12, 4, and 18 miles.

27. A commission merchant receives the sum of s dollars to invest in merchandise—himself to retain a commission of r per cent. on the amount of the purchase. What is the sum to be invested?

Ans. $\frac{100s}{100+r}$.

28. A sum of money was equally divided among a number of persons, by first giving to A $100 and $\frac{1}{6}$ of the remainder, then to B $200 and $\frac{1}{6}$ of the remainder, then to C $300 and $\frac{1}{6}$ of the remainder; and so on. What was the sum divided? and the number of persons?

Ans. $2500, and 5 persons.

29. The sum of $500 is to be applied in part towards the payment of a debt of $900, now due, and in part to paying the interest, at 7 per cent., in advance, on the remainder of the debt, on which a credit of 12 months is to be allowed. What is the amount of payment that can be made on the debt? *Ans.* 469.89'

30. A besieged garrison had such a quantity of bread as would, if distributed to each man at 10 ounces a day, last 6 weeks; but having lost 1200 men in a sally, the governor was enabled to increase the allowance to 12 ounces per day. What was the original number of men?

Ans. 7200 men.

31. A is indebted to B the sum of s dollars, and is able to raise but a dollars. With this latter sum A proposes to pay a part of the debt, and the interest, at r per cent., in advance, on his note at n years, for the remainder. For what sum should the note be drawn.

Ans. $\frac{100\,(s-a)}{100-rn}$.

32. The crew of a ship consisted of her complement of sailors and a number of soldiers. Now there were 22 sailors to every three guns, and 10 over. Also the whole number of men was 5 times the number of soldiers and guns together. But after an engagement, in which the slain were $\frac{1}{4}$ of the survivors, there wanted 5 of being 13 men to every 2 guns. Required the number of guns, soldiers, and sailors. *Ans.* 90 guns, 55 soldiers, 670 sailors.

33. Two sums of money, amounting together to $600, were put at interest—the smaller at 2 per cent. more than the other. The interest of the larger sum was afterwards increased, and that of the smaller diminished, 1 per cent. By this the interest of the whole was augmented *one-twentieth.* But if the interest of the greater sum had been so increased, without any diminution of the other, the interest of the whole would have been increased *one-tenth.* What were the two sums? and the two rates of interest?

Ans. $400 at 6 per cent.; $200 at 8 per cent.

34. Two persons purchase 300 acres of land at $2 per acre, each one paying $300; but the first takes the more fertile portion of the

tract, at 25 cents above the mean price per acre, and the second the remainder at 25 cents below the mean price per acre. How many acres has each?

Ans. This problem is *impossible;* how is its impossibility indicated?

35. The *area* of a field is 432 square rods, and the sum of its length and breadth is equal to twice their difference. Required the length and breadth of the field. *Ans*. 36, and 12 rods.

36. A and B lay out some money on speculation. A disposes of his interest in the business for £11, and gains as much per cent. as B lays out; B gains £36, and it appears that A gains 4 times as much per cent. as B. What was the capital of each? *Ans*. £5, and £120.

37. A garden which is 12 rods in length, and 8 rods in breadth, is surrounded by a walk whose area is equal to $\frac{1}{6}$ of the area of the garden itself. Required the breadth of the walk. *Ans*. $-5+\sqrt{29}$.

38. A and B hired a pasture, into which A put 4 horses, and B as many as cost him 18 shillings a week. Afterwards B put in two additional horses, and found that he must pay 20 shillings a week. At what rate was the pasture hired? *Ans*. 30*s*. per week.

39. A gentleman bought a rectangular piece of ground, at $10 for every rod in its *perimeter*. If the same area had been in the form of a square, and had been purchased in the same way, it would have cost $20 less; and a square piece of the same perimeter would have contained $12\frac{1}{4}$ square rods more. What were the length and breadth of the lot? *Ans*. 16, and 9 rods.

40. A person being asked the ages of himself and his wife, replied, that the product of their ages added to the square of his age, would make 1560, but added to the square of hers would make 1144. What were their ages? *Ans*. 30, and 22.

41. A and B purchased a farm containing 900 acres for which they paid $900 each. On dividing the land, it was agreed that A should have his choice of situation, and pay 45 cents per acre more than B. How many acres should each have taken? and at what price per acre?

Ans. A 400 acres at $2.25; B 500 acres at $1.80.

42. A capital of $13,000 was divided into two parts, which were put at interest in such a manner that the income was the same from each. If the first part had been at the same rate of interest as the second, it would have produced an income of $360; and if the second part had been at the same rate as the first, it would have produced an income of $490. What were the two rates of interest?

Ans. 7 and 6 per cent.

43. A departs from London towards Lincoln at the same time at which B leaves Lincoln for London. When they met, A had traveled 20 miles more than B, having gone as far in $6\frac{2}{3}$ days as B had in all the time; and it appeared that B would not reach London under 15

days. What is the distance between the two places? and how far had each man traveled?

Ans. Distance, 100 miles; A had gone 60, B 40 miles.

44. The product of the two dimensions of a rectangular piece of land, subtracted from the square of the greater dimension, leaves 300 square rods, and subtracted from the square of the less, leaves 200 square rods. What are the dimensions of the piece?

Ans. This problem is *impossible;* how is its impossibility indicated? and in what does this impossibility consist?

45. A person being asked the ages of his two children, replied, that the difference of their ages was 3 years, and the product multiplied by the sum of their ages was 308. What were their ages?

This problem will result in an Affected Cubic Equation.

Ans. 7 and 4 years.

46. A gentleman who had a square lot of ground, reserved 10 square rods out of it, and sold the remainder for $432, which was as many dollars per square rod as there were rods in a side of the whole square? What was the length of its sides? *Ans.* 8 rods.

47. A and B set out together from the same place, and travel in the same direction. A goes the first day 28 miles, the second 26, and so on, in arithmetical progression; while B goes uniformly 20 miles per day. In how many days will the two be together again?

Ans. 9 days.

48. A farmer wishes to build a crib whose capacity shall be 1620 cubic feet, and whose length, breadth, and height shall be in an arithmetical progression decreasing by the common difference 3. What must be the dimensions of the crib?

It may be well to remind the student here, that *cubic measure,* or measure of capacity, is found by multiplying together *length, breadth,* and *height* or depth. *Ans.* 15, 12, and 9 feet.

49. One traveler sets out to go from A to B, at the same time at which another sets out from B to A. They both travel uniformly, and at such rates, that the former, 4 hours after their meeting, arrives at B, and the latter at A, in 9 hours after. In how many hours did each one perform the journey? *Ans.* 10, and 15 hours.

50. A lady on being asked the ages of her three little boys, answered that they were in harmonical progression; and the sum of their ages was 22 years; and that if the ages of the two elder were each increased by $\frac{1}{3}$ of itself, the three would then be in geometrical progression. What were the respective ages? *Ans.* 4, 6, and 12 years.

51. A person wishes to construct two cubical reservoirs which shall differ in their linear dimensions by 4 feet, and which shall together contain 5824 cubic feet. What must be the dimensions of the two reservoirs? *Ans.* 12, and 16 feet

52. Two partners, A and B, divided their gain, which was $60, when B's share was found to be $20. A's capital was in trade 4 months; and if the number 50 be divided by A's capital, the quotient will be the number of months that B's capital, which was $100, continued in trade. What was A's capital? and the time B's was in trade? *Ans.* A's Capital $50; B's 1 month in trade.

53. Let there be a square whose side is 110 inches; it is required to assign the length and breadth of a rectangle whose perimeter shall be greater than that of the square by 4 inches, but whose area shall be less than the area of the square by 4 square inches.

Ans. 126, and 96 inches.

54. There is a number consisting of three digits which increase from left to right by the common difference 2; and the product of the three digits is 105. Required the number. *Ans.* 357.

55. A person bought 2 pieces of cloth for $63. For the first piece he paid as many dollars per yard as there were yards in both pieces, and for the second as many dollars per yard as there were yards in the first more than in the second; also the first piece cost six times as much as the second. What was the number of yards in each piece?

Ans. 6, and 3 yards.

56. There is a number consisting of 4 digits which decrease from left to right by the common difference 2; and the product of the four digits is 945. Required the number.

Ans. 9753.

57. A gentleman purchased two square lots of ground for $300; each of them cost as many cents per square rod as there were rods in a side of the other, and the greater contained 500 square rods more than the less. What was the cost of each lot?

Ans. $180, and $120.

58. A merchant bought a number of bales of cloth. The number of pieces in each bale was 10 more than the number of bales, and the number of yards in each piece was 5 more than the number of pieces in each bale; and the whole quantity was 1500 yards. What was the number of bales? *Ans.* 5 bales.

59. A person dies, leaving children, and a fortune of $46800, which, by his will, is to be divided equally amongst them. Immediately after the death of the father, two of the children also die, in consequence of which each surviving one receives $1950 more than he was entitled to by the will. How many children did the father leave?

Ans. 8 children.

60. A coach set out from Cambridge for London with 4 more *outside* than *inside* passengers. Seven outside passengers went at 2 shillings less than 4 inside ones, and the fare of the whole amounted to £9. At the end of half the journey, 3 more outside and one more inside passenger were taken up, in consequence of which the fare of

the whole was increased in the proportion of 19 to 15. Required the number of passengers at first, and the fare of each.

Ans. 5 inside, and 9 outside passengers; fares 18 and 10 shillings.

61. There is a fraction which, inverted and increased by $\frac{1}{3}$, will be less than 2; and if its numerator be increased by 2, the value of the fraction will be greater than $\frac{1}{2}$. What is the fraction?

Ans. Between $\frac{3}{5}$ and $\frac{1}{6}$.

62. In a purse which contains 24 coins of silver and copper, each silver coin is worth as many pence as there are copper coins, each copper coin is worth as many pence as there are silver coins, and the whole is worth 18 shillings. How many were there of each kind of coins?

Ans. 6, and 18.

63. The square root of a certain number plus 4, is less than 9; and ten times the square root of the number minus 2, is greater than eight times the square root of the number plus 4. What is the number?

Ans. Between 9 and 25.

64. A and B traveled on the same road, and at the same rate, from Huntingdon to London. At the 50th mile stone from London, A overtook a drove of geese, which were proceeding at the rate of 3 miles in 2 hours; and 2 hours afterwards met a wagon which was moving at the rate of 9 miles in 4 hours. B overtook the same drove of geese at the 45th mile stone, and met the same wagon 40 minutes before he came to the 31st mile stone. Where was B when A reached London?

Ans. 25 miles from London.

CHAPTER XII.

GENERAL THEORY OF EQUATIONS.

STURM'S THEOREM—HORNER'S METHOD OF SOLVING THE HIGHER EQUATIONS.

(280.) A GENERAL THEORY OF EQUATIONS consists of an exposition of the general properties of Equations.

Properties relating to the *Divisors* of an equation (253), a criterion for *Roots* of an equation (254), the *Number of roots* (255), and the general *Law of the Coefficients* of an equation (267), have been presented in a former Chapter; and it is only necessary therefore to add here the more important of the remaining parts of this extensive theory.

General Equation.

(281.) Every complete Equation containing only integral powers of the unknown quantity x, may be reduced to the general form

$$x^n+ax^{n-1}+bx^{n-2}+\ldots+k=0;$$

by transposing all the terms to one side, and dividing by the coefficient of the highest power of x.

The exponent n is equal to the degree of the particular Equation, and the exponents decrease regularly by 1 in the successive terms.

If any of the lower powers of x be wanting in the given Equation, those powers may be introduced by giving to each of them the coefficient 0—the Equation will thus be rendered *complete.*

Thus the Equation $x^3+4x-10=0$, when made complete is,

$$x^3+0x^2+4x-10=0,$$

or, simply, $x^3+0\quad+4x-10=0$.

Some of the terms must necessarily be *positive*, and others *negative*, in the first member, to make the second member 0; but in the general Equation the sign + is prefixed to each term, to denote the algebraic sum of all the terms.

Change of Signs of the Roots of Equations.

(282.) If the signs of the alternate terms of any complete **Equation** be changed, the signs of *all the roots* will be changed.

If we take the general Equation,

$$(1)\qquad x^n+ax^{n-1}+bx^{n-2}+\text{ \&c.}=0;$$

and change the signs of the alternate terms, we shall have

$$(2)\qquad x^n-ax^{n-1}+bx^{n-2}-\text{\&c.}=0,\text{ or}$$
$$(3)\qquad 0=-x^n+ax^{n-1}-bx^{n-2}+\text{\&c.}$$

When the exponent n is an *even* number, the terms containing the *odd* powers of x will be *negative* in Equation (2); and when n is an *odd* number, the terms containing the odd powers of x will be negative in Equation (3).

If now the substitution of r for x in Equation (1) result in 0, the same result will be obtained by substituting $-r$ for x in Equation (2) or (3); since the odd *negative* powers of $-r$, (202), in the latter Equations will become *positive* when taken subtractively, and these Equations will then be the same as Equation (1).

Hence, if r be a root of Equation (1); $-r$ will be a root of Equation (2) or (3); that is, if the signs of the alternate terms, &c.

The roots of the Equation

$$x^3-7x^2+36=(x-6)\,(x-3)\,(x+2)=0,$$

are 6, 3, and -2, (254); what is the Equation whose roots are

$$-6,\ -3,\text{ and } 2\,?$$

Completing the given Equation by introducing the first power of x, and changing the signs of the alternate terms, we have

$$x^3+7x^2+0x-36\text{ or }x^3+7x^2-36=(x+6)\,(x+3)\,(x-2)=0.$$

of which the roots are -6, -3, and 2, (254).

From the principle thus demonstrated it follows, that

(283.) The *positive* roots of any complete Equation become ***negative***, **and the negative ones become positive, by changing the signs of the alternate terms of the Equation.**

Functions of Variable Quantities.

(284.) A *Function* of a variable quantity is any algebraic expression containing that quantity.

Thus, if x be a variable quantity, that is, a symbol to which values may be attributed at pleasure, then $2x+a$, $3x^2$, x^3-ax^2+bx, &c., are different *functions* of x.

Derivative Functions.

(285.) The *Derivatives* of a Function of x are the *coefficients* of the factors

$$h,\ \frac{h^2}{2},\ \frac{h^3}{2\times3},\ \frac{h^4}{2\times3\times4},\ \&c.$$

in the development of the Function, when $x+h$ is substituted for x.

The *first derivative* consists of the coefficients of h; the *second* of the coefficients of $\frac{h^2}{2}$; the *third* of the coefficients of $\frac{h^3}{2\times3}$, &c.

If we take the general Function

$$x^n+ax^{n-1}+bx^{n-2}+\ \&c.,$$

and substitute $x+h$ for x, we shall have

$$(x+h)^n+a(x+h)^{n-1}+b(x+h)^{n-2}+\ \&c.$$

which—by developing the different powers of $(x+h)$—collecting the terms which are independent of h—those which are coefficients of h,—those which are coefficients of $\frac{1}{2}h^2$, &c., becomes

$$\begin{array}{l} x^n+ax^{n-1}+bx^{n-2}+\&c. \\ +(nx^{n-1}+(n-1)ax^{n-2}+(n-2)bx^{n-3}+\&c.)h \\ +(n(n-1)x^{n-2}+(n-1)\,(n-2)ax^{n-3}+(n-2)\,(n-3)bx^{n-4}+\&c.)\,\frac{h^2}{2} \\ \qquad\&c. \qquad\qquad \&c. \qquad\qquad \&c. \end{array}$$

In this development, the first horizontal line is the same as the original Function; the polynomial coefficient of h in the second line, is the *first derivative* of the Function; the polynomial coefficient of $\frac{1}{2}h^2$ in the third line, is the *second derivative* of the Function; &c.

If therefore X be taken to represent the given Function; D', D'', &c., its successive *derivatives*; and X' the Function after $x+h$ has been substituted for x, we shall have

$$X'=X+D'h+D''\frac{h^2}{2}+\ \&c.$$

Law of Derivative Functions.

(286.) The *first* Derivative of a given Function of x will be found by multiplying each term of the function by the exponent of x in the term, and diminishing the exponent by *unity.*

In like manner the *second* Derivative may be obtained from the first; the *third* from the second, &c.

This law of derivation is deducible from the preceding general example of a function and its derivatives.

For a particular example take the Function

$$x^3-4ax^2+3a^2x-a^3.$$

According to the law above stated, the successive Derivatives are

$$D' = 3x^2-8ax+3a^2;$$
$$D'' = 6x-8a;$$
$$D''' = 6.$$

No other derivative can be obtained, since the last one, namely, 6, is not a function of x; in other words, the fourth derivative is 0.

A term which does not contain x, as a^3, $3a^2$ and $8a$ in this example, disappears in the derivation, since x may be understood in such terms with exponent 0.

1. What is the first Derivative of the function

$$2x^3+5x^2-5x+10?$$

Ans. $6x^2+10x-5$.

2. What is the second Derivative of the function

$$x^4-2x^3+3x^2-4x?$$

Ans. $12x^2-12x+6$.

3. What is the third Derivative of the function

$$x^5-4x^4+3x^3-5x^2+6x?$$

Ans. $60x^2-96x+18$.

When the simple term *derivative* is used hereafter, it must be understood to mean the *first* derivative of the Function.

Derivatives of Composite Functions.

(287.) The Derivative of the *product* of two different Functions of x, is equal to the sum of the products which arise from multiplying each function into the derivative of the other function.

Let $x+h$ be substituted for x in each of the two functions; then, as already explained, we shall have for the development of one of the functions,

$$X'=X+D'h+\text{ \&c., (285).}$$

By employing a corresponding notation, and proceeding in the same manner with the other function, we shall have

$$Y'=Y+R'h+\text{ \&c.}$$

in which Y represents the given Function, R′ its first derivative, and Y′ the function after $x+h$ has been substituted for x.

Multiplying together the corresponding members of these equations, we have for the product of the two functions, after $x+h$ has been substituted for x in each,

$$X'Y'=XY+(YD'+XR')h+\text{ \&c.}$$

The polynomial coefficient of h in this development, is the first Derivative of the product of the two functions; and it is the sum of the products obtained by multiplying each of the functions X and Y into the derivative of the other function.

The Derivative of the product $(x-1)^3(x-2)^2$ is

$$(x-1)^3.\,2\,(x-2)+(x-2)^2.\,3\,(x-1)^2.$$

In this example the derivatives of the binomials are obtained by multiplying each binomial by its exponent, and diminishing the exponent by *unity*, in the same manner as for monomial terms (286).

Thus the derivative of $(x-1)^3$ is $3(x-1)^2$;
and the derivative of $(x-2)^2$ is $2(x-2)$.

The principles of derivation are involved in the following method of finding the

Equal Roots of Equations.

(288.) When an Equation has *equal roots*, such roots will be found by solving the equation which is formed by putting the greatest *Common Measure* of the given equation and its derivative, *equal to* 0.

If there be no such common measure, the given Equation has no equal roots.

Let us take the Equation

$$(x-a)^n=0,$$

which has n roots each equal to a, since it admits of n successive divisions by $x-a$ (254).

The Derivative of the first member is $n(x-a)^{n-1}$, (286); and $(x-a)^{n-1}$ is evidently the greatest *common measure* of the Equation and its Derivative.

By putting this common measure equal to 0, we have

$$(x-a)^{n-1}=0,$$

which has $n-1$ roots each equal to a; and hence the number of equal roots of the given Equation is one more than of the new equation.

In further illustration of this subject, take the Equation

$$(x-a)^3\,(x+b)^2=0,$$

which has 3 roots each equal to a, and 2 roots each equal to $-b$ (254).

The Derivative of the first member is

$$(x-a)^3.\ 2\,(x+b)+(x+b)^2.\ 3\,(x-a)^2 \text{ (287)};$$

and the greatest *common measure* of this Derivative and the given Equation is evidently $(x-a)^2(x+b)$.

By putting this common measure equal to 0, we have

$$(x-a)^2\,(x+b)=0,$$

which has 2 roots each equal to a, and one $=-b$; the number of each *set of equal roots* of the given Equation being one more than of the new equation.

The number of *equal roots*, or of *each set* of equal roots, of the given Equation, will always be one more than of the new equation; because each of the binomial factors corresponding to the equal roots, is in *a power one degree higher* in the given than in the new equation.

When the given equation involves only the *first power* of each of the binomial factors corresponding to its roots, these factors will disappear in its Derivative,—their exponents becoming 0 (286). In this case the Equation will have no equal roots; and will evidently have no *common measure* with its derivative.

(289.) If an Equation be divided by the product of all the binomial factors corresponding to its *equal roots;* the Quotient put $=0$, will form an Equation whose roots are the *remaining roots* of the given Equation.

This is evident from considering, that the first member of an Equation whose second member is 0, is the product of the binomial factors corresponding to *all the roots* of the Equation.

As an application of the preceding principles, let it be required to find the roots of the Equation

$$x^5+x^4-9x^3-5x^2+16x+12=0.$$

The Derivative of this Equation is

$$5x^4+4x^3-27x^2-10x+16, \quad (286).$$

The greatest *common measure* of the Equation and its derivative will be found to be x^2-x-2, (66).

The two roots of the Equation

$$x^2-x-2=0, \text{ or } x^2-x=2,$$

will be found to be 2 and -1.

The given Equation has therefore *two roots*, each equal to 2, and *two*, each equal to -1.

The product of the factors $x-2$, $x-2$, $x+1$, $x+1$, which correspond to the equal roots, is

$$x^4-2x^3-3x^2+4x+4.$$

The given Equation, divided by this product, gives

$$x+3=0, \text{ or } x=-3.$$

The *five roots* are therefore 2, 2, -1, -1, and -3.

1. Find the several roots of the Equation

$$x^3-3x^2+4=0.$$

Ans. 2, 2, and -1.

2. Find the several roots of the Equation

$$x^3+4x^2-3x-18=0.$$

Ans. 2, -3, and -3.

3. Find the several roots of the Equation

$$x^4-4x^3-2x^2+12x+9=0.$$

Ans. -1, -1, 3, and 3

Real Roots of Equations.

The *real* roots of an Equation are so called to distinguish them from *imaginary* quantities (246), which are sometimes found among the values of x. Real roots may be either positive or *negative*, rational or *irrational*.

(290.) If an Equation whose second member is 0, give results with *contrary signs* when any two numbers are substituted for the unknown quantity, the Equation will have at least *one real root* between those numbers.

If in the Equation

$$x^3-3x^2+5x-36=0,$$

we substitute 5 for x, the result will be

$$125-75+25-36=39:$$

and if we substitute 2 for x, the result will be

$$8-12+10-36=-30.$$

If the value of x vary continually, the value of the first member of the Equation will vary continually. But when x thus varies from 5 to 2, the first member passes through all the values between 39 and -30; and must therefore pass through 0.

There is therefore some number between 5 and 2, which, substituted for x, will make the first member $=0$, and that number is a root of the Equation.

(291.) *Cor.* If the two numbers which give results with *contrary signs*, when substituted for x, differ by *unity;* the smaller number (regarding both as *positive*), is the *integral* part of a root of the Equation.

In the Equation $x^4-12x^3-20x^2-36=0$,

if 13 and 14 be substituted for x, the results will be -1219 and 1532 respectively; hence, 13 is the integral part of a root of the equation.

Again; if -2 and -3 be substituted for x, in the same equation, the results will be -4 and 189 respectively; hence, -2 is the integral part of another root of the equation.

Irrational and Imaginary Roots.

(292.) *Irrational* and *imaginary Roots* enter Equations by pairs; so that an Equation will have an *even number* of such roots, or none at all.

For, if the Equation be divided by the product of the binomial factors corresponding to all of its roots but *two,* the quotient put $=0$, will form a *quadratic* Equation.

Let $a+\sqrt{b}$ be one of the roots of this quadratic; then will $a-\sqrt{b}$ be the other root, since the square root of b is $\pm\sqrt{b}$, and the two roots, $a\pm\sqrt{b}$, are evidently the two roots of the given equation which were excluded from the preceding *product.* If therefore an equation contain one irrational root, $a+\sqrt{b}$, it will also contain another, $a-\sqrt{b}$.

In a similar manner, it may be shown, that if any *odd* number of the roots be irrational, there will be also another irrational root;—and the demonstration, it is evident, will be equally applicable to *imaginary roots.*

(293.) *Cor.* An Equation of an *even degree,* may have all its roots *imaginary;* but if they are not all so, two of them at least are *real.*

The precise number of *real,* and thence of *imaginary* roots of an Equation, is determined by the following Theorem of STURM—preliminary to which is the definition of

Variations and Permanences of Signs.

(294.) Two consecutive signs in a series of terms, constitute a *variation* when one of them is + and the other —, and a *permanence* when they are both + or both —.

Thus, in the polynomial

$$a-b+c+x-y-z,$$

the signs of a and b constitute a *variation,* of b and c a variation, of c and x a *permanence,* of x and y a variation, and of y and z a permanence

In the given polynomial there are therefore 3 *variations* and 2 *permanences* among the signs of the terms.

STURM'S THEOREM.

(295.) This Theorem, which bears the name of its discoverer, explains a general method of determining the number of *real* roots, and among them the number of *positive* and *negative* roots of any given Equation.

If the Equation has two or more *equal roots,* these may be found by the method already explained (288); and it is necessary, therefore, to consider only the general case in which the roots are *unequal.*

The statement of the Theorem involves several particulars.

1. Let the Equation, supposed to have *no equal roots,*—with all its terms on one side,—be represented by

$$X=0\ ;$$

find the first Derivative of X, and denote it by D' (286).

2. Divide X by D', until the remainder is of a lower degree than the divisor, and denote the remainder, with its *signs changed,* by R'. Divide D' by R', in the same manner, and denote the remainder, with its *signs changed,* by R''; and so on, as in finding the greatest common measure, until the remainder is independent of x; and denote this last remainder, with its sign changed, by K.

3. Find the *row of signs* of the values of the Functions

$$X,\ D',\ R',\ R'',\ \ldots.\ K$$

which result from substituting any number p for x, and also from substituting any other number t for x; then—

The difference between the number of *variations* in the first *row of signs* and the number of *variations* in the second, will be the number of *real* roots of the Equation comprised between the numbers p and t.

The *whole number of the real roots* of the Equation, will thus be determined, if p and t be taken equal to $+\infty$ and $-\infty$, since all real numbers are included between these limits.

The number of *positive* roots will be determined by taking p and t equal to $+\infty$ and 0; and the number of *negative* roots, by taking p and t equal to 0 and $-\infty$.

We will first apply, and then demonstrate this Theorem.

EXAMPLE I.

To determine the number of *real* roots of the Equation

$$x^3-3x^2+5x+7=0,$$

which is known to have *no equal roots.*

The derivative of the first member is $3x^2-6x+5$ (286); and having now the two Functions which correspond to X and D', we must find R', &c., according to the Theorem.

To avoid *fractions* in the several divisions, we may multiply or divide the Functions by *positive* numbers, as in finding the greatest common measure, since this will not affect the *signs* of the Functions; but in this example, the last division will be performed *fractionally.*

We have then $X = x^3-3x^2+5x+7$;
$D'=3x^2-6x+5$;
and shall find $R'=-2x-13$;
$K=-341\frac{1}{2}$.

If we substitute ∞, *infinity*, for x in these Functions, it is evident that the first term of each polynomial will be greater than the sum of all the other terms; so that the *sign of each result* will be the same as that of its *first term.*

	X	D'	R'	K	
$-\infty$ for x gives the *signs*	$-$	$+$	$+$	$-$,	2 *variations;*
∞ for x gives the *signs*	$+$	$+$	$-$	$-$,	1 *variation.*

The difference between the number of *variations* in the first row of signs, and the number in the second, is *one;* the given Equation has then but *one real root,* and has consequently *two imaginary roots* (255)

To determine whether the real root is *positive* or *negative.*

	X	D'	R'	K	
0 for x gives the *signs*	$+$	$+$	$-$	$-$,	1 *variation;*

and since we have already seen that $-\infty$ for x, gives 2 *variations,* the real root is comprised between 0 and $-\infty$, which shows that this root is *negative.*

Demonstration of Sturm's Theorem.

The demonstration of this Theorem depends on several distinct considerations respecting the series of Functions which it involves.

I. Two *consecutive Functions* in the series X, D', R', R'',...K, cannot both vanish, that is, become 0, for the same value of x.

For, denoting by q', q'', &c., the *quotients* of the successive divisions prescribed in the Theorem; then R', R'', &c., denoting the successive *remainders*, with their *signs changed*, we have

$$\begin{array}{ll} (1) & X = D'q' - R' \\ (2) & D' = R'q'' - R'' \\ (3) & R' = R''q''' - R''', \text{ \&c.} \end{array}$$

If now $D'=0$, and $R'=0$, for the same value of x, Equation (2) would give $R''=0$; and then Equation (3) would give $R'''=0$; and so on, to the last remainder $K=0$. But K is independent of x, and cannot therefore reduce to 0 for any value of x.

II. If any one of the Functions between X and K vanish for any value of x, the two *adjacent* Functions will have *contrary signs.*

Thus, if $R'=0$, Equation (2) will give $D'=-R''$;
that is, the two Functions in the series which are adjacent to the vanishing one, will have equal values but *contrary signs.*

III. The number of *variations of signs* of the whole series, X, D', R', R'', &c., cannot be affected by the vanishing, or change of signs, of any of the Functions between X and K, for any value of x.

For no two *consecutive Functions* can vanish at the same time (I); and if any two or more which are not consecutive, could vanish together; each vanishing Function, being interjacent to two others which have *contrary signs* (II), would form a variation with one, and only one, of them before vanishing, while those two would form a variation with each other afterwards.

Again, since a Function can change its sign only by *passing through* 0, two consecutive ones cannot change their signs together (I); and when they are not consecutive, their changes of signs will not affect the whole number of variations.

IV. If r be a root of the Equation $X=0$, the signs of X and D' will form a *variation* for a value of x which is a little less than r, and a *permanence* for a value of x which is a little greater than r.

Since r is supposed $=x$ in the Equation $X=0$, the substitution of $r+h$ for x will result in the same development as the substitution of $x+h$, in the Function X. This development, we have heretofore seen, is

$$X+D'h+D''\frac{h^2}{2}+\&c. \ (285).$$

But $X=0$, while D' is not $=0$ (I); we have therefore for the development of the Function X,

$$D'h+D''\frac{h^2}{2}+\&c.$$

By substituting $r+h$ for x in the Function D', we should, in like manner, find the development of this Function to be

$$D'+D''h+D'''\frac{h}{2}+\&c.,$$

in which D'' and D''' are the *first* and *second derivatives* of D'.

Now h may be taken so small, that the *first term* in each of the last two polynomials, shall be greater than the sum of all the other terms in each—the effect of multiplying by a *proper fraction* being, to *diminish the quantity multiplied.*

The sign of each of the two polynomials will then be the same as that of its first term, $D'h$ and D'. .

They will therefore have the *same* sign when h is *positive*, and *contrary* signs when h is *negative;* that is, the signs of the two Functions, X and D', form a *variation* for $x=r-h$, and a *permanence* for $x=r+h$.

V. Let the number p be less than t, and let it *increase continually*, until it becomes equal to t; then, if p be substituted for x in the series of Functions

$$X, D', R', R'' \ldots K,$$

the number of *variations* among the *signs* of the results cannot be affected by the vanishing, or change of signs, of any of the Functions between X and K (III); and the Function K can in no way be affected by the substitution, since this Function is entirely independent of x.

But when p increases from a number which is a little *less*, to one which is a little *greater*, than a root of the Equation $X=0$, the signs of X and D' will be changed from a *variation* to a *permanence* (IV).

One variation of signs will thus be lost whenever the increasing values of p, pass a root of the given Equation; and therefore the number of *variations lost* when the substituted numbers increase from p to t, is equal to the number of *real roots* of the given Equation comprised between the numbers p and t.

Thus the Theorem is demonstrated.

EXAMPLE II.

To determine the number of the real roots of the Equation

$$x^3-2x^2-5x+6=0,$$

which is known to have *no equal roots.*

The series of Functions will be found to be

$$\begin{aligned} X &= x^3-2x^2-5x+6 \\ D' &= 3x^2-4x-5 \\ R' &= 19x-22 \\ K &= 106\tfrac{11}{19} \end{aligned}$$

All the real roots of the Equation are comprised between $-\infty$ and $+\infty$; and when *infinity* is substituted for x, it is evident that the result in each polynomial will have the same sign as the result in the *first term* of the polynomial.

	X	D'	R'	K	
$-\infty$ for x gives	$-$	$+$	$-$	$+$	, 3 *variations;*
$+\infty$ for x gives	$+$	$+$	$+$	$+$	, 0 *variation.*

The three roots of the given Equation are therefore all *real.*

	X	D'	R'	K	
0 for x gives	$+$	$-$	$-$	$+$	, 2 *variations.*

One of the roots is thus found between $-\infty$ and 0, and is therefore *negative;* while the other two are between 0 and $+\infty$, and are consequently *positive.*

Limits and Situation of the Real Roots of Equations.

(296.) The *limits* to a root of an Equation, are two numbers, one *greater*, and the other *less*, than the root; and the root is said to be *situated* between those limits. When the two numbers differ from each other by *unity*, they are the *nearest integral limits* of the root.

The limits of all the *positive* roots will be found by substituting for x in the series of Functions

$$X, D', R', R'', \cdots K,$$

the values, ∞, 0, 1, 2, 3, &c., until the row of signs becomes the same that results from substituting ∞; and by observing where *one or more variations are lost*, the situation of each of these roots will be determined by its nearest integral limits.

The limits and situation of the *negative* roots may be found, in like manner, by substituting the values $-\infty$, 0, -1, -2, -3, &c.

Thus, to recur to the preceding Example, we shall find that

		X	D'	R'	K	
∞	for x gives	+	+	+	+	, 0 *variation;*
0	" gives	+	−	−	+	, 2 *variations;*
1	" gives	0	−	−	+	
2	" gives	−	−	+	+	, 1 *variation;*
3	" gives	0	+	+	+	
4	" gives	+	+	+	+	, 0 *variation;*
$-\infty$	" gives	−	+	−	+	, 3 *variations;*
-1	" gives	+	+	−	+	, 2 *variations;*
-2	" gives	0	+	−	+	
-3	" gives	−	+	−	+	, 3 *variations.*

Here, it will be seen that ∞ and 4, when substituted for x, produce the same row of signs; all the *positive* roots are therefore comprised between 0 and 4.

Also $-\infty$ and -3 produce the same row of signs; which shows that all the *negative* roots are included between 0 and -3.

The substitution of 1, 3, or -2, reduces the Function X to 0; these numbers are the roots of the given Equation.

It will also be noticed, that in passing these roots, the Function X changes its sign, and we should know that the Equation has at least *one real root* between each of the two numbers which produce a change of sign in the Function X (290), even if the intermediate substitutions had not been made.

1. How many *real* roots has the Equation

$$x^3-5x^2+8x-1=0,$$

which is known to have *no equal roots?* How many *imaginary* roots has the same equation?

Ans. 1 real root; and two imaginary roots.

2. How many real roots has the Equation

$$x^3-7x-6=0,$$

which is known to have no equal roots? How many of these roots are *positive?* and how many *negative?*

Ans. 3 real roots; 1 positive, and 2 negative.

3. How many real roots has the Equation

$$x^3-7x+7=0,$$

which is known to have no equal roots? How many of these roots are positive? and how many negative? What are the nearest *integral* limits of each root?

Ans. 3 real roots; 2 positive roots between 1 and 2, and 1 negative root between -3 and -4.

HORNER'S METHOD

Of Solving the Higher Equations.

(297.) The general principles of this method, which bears the name of its discoverer, and which is much admired for its simplicity and elegance, may be elucidated as follows:

1. Let any *quadratic* Equation be represented by

$$(1)\qquad Ax^2+Bx=N.$$

Let the *integral* part, found by trial (291), of one of its positive roots be denoted by r, and the *decimal* part by y; so that $x=r+y$, or $y=x-r$.

If now $r+y$ be substituted for x in the given Equation, that Equation will be *transformed* into another, of the same degree, in which the values of y will be less by r than the values of x in the given Equation.

Making the substitution, we have

$$A(r+y)^2+B(r+y)=N;$$

which, by developing, becomes

$$Ar^2+2Ary+Ay^2+Br+By=N;$$

or, by arranging the unknown, and transposing the known terms,

$$(2)\qquad Ay^2+(B+2Ar)y=N-(Br+Ar^2).$$

Since y represents the *decimal* part of the root of the given Equation, y^2 is of small value in comparison with y itself; and we should probably find the *first figure* in the value of y—which would be the *first decimal* figure in the root of the given Equation—by omitting the term in y^2, and taking

$$y=\frac{N-(Br+Ar^2)}{B+2Ar}.$$

Denoting the first figure in the value of y by d', and substituting $d'+z$ for y, in Equation (2), that Equation will be *transformed* into another, in which the values of z will be less by d' than those of y in Equation (2).

The value of z, will commence in the *second* decimal place, and z^2 will be very small in comparison with z; so that we should pretty certainly find the first figure in the value of z—which would be the *second decimal* figure in the root of Equation (1)—by dividing the second member of this last equation by the *coefficient* of z.

We might go on, in like manner, to find any number of decimal figures in the root of the given equation.

2. Let any *cubic* Equation be represented by

$$(1) \qquad Ax^3+Bx^2+Cx=N.$$

Let the *integral* part found by trial (291), of one of its positive roots, be denoted by r, and the *decimal* part by y; so that $x=r+y$, or $y=x-r$.

By substituting $r+y$ for x, we have

$$A(r+y)^3+B(r+y)^2+C(r+y)=N;$$

which, by developing, arranging the unknown, and transposing the known terms, becomes

$$(2) \quad Ay^3+(B+3Ar)y^2+(C+2Br+3Ar^2)y=N-(Cr+Br^2+Ar^3).$$

y^3 and y^2 are of small value in comparison with y; and we should probably find the *first figure* in the value of y—which would be the *first decimal* figure in the root of equation (1)—by omitting the terms in y^3 and y^2, and taking

$$y=\frac{N-(Cr+Br^2+Ar^3)}{C+2Br+3Ar^2}$$

By considering the following arrangement of quantities, we may discover a method by which equation (2), may be conveniently obtained from equation (1)—*without the tedious process of substitution.*

$$\begin{array}{llrrl}
A & & +B & +C & =N \\
 & & Ar & Br+Ar^2 & \quad Cr+Br^2+Ar^3 \\
 & & \overline{B+Ar} & \overline{C+Br+Ar^2} & \\
 & & Ar & Br+2Ar^2 & \\
 & & \overline{B+2Ar} & \overline{C+2Br+3Ar^2} & \\
 & & Ar & & \\
 & & \overline{B+3Ar} & &
\end{array}$$

The trinomial under N, subtracted from N, gives the second member of Equation (2); the last trinomial under C, is the coefficient of y; the last binomial under B, is the coefficient of y^2; and A is the coefficient of y^3, the same as of x^3, in Equation (1).

From these coefficients, and the second member, those in the next transformed or derived equation would be obtained, in the same manner in which these are obtained from Equation (1); and we should thence, in like manner, find the 2d *decimal* figure in the root of Equation (1); and so on.

The preceding method, applied to an equation of any of the higher degrees, results in the following

RULE XXV.

(298.) *To find the Real Roots of the Higher Equations.*

A, *B*, *C*, &c., denoting the coefficients of x^n, x^{n-1}, x^{n-2}, &c.; *N*, the second member; and *r*, the *integral* part, found by trial (291), of any *positive* root of the Equation ;—

1. Multiply *A* by *r*, and add the product to *B;* multiply this sum by *r*, and add the product to *C;* and so on, to the coefficient of *x;* multiply the last sum by *r*, and *subtract* the product from *N*.

The *remainder* will be the *second member* of a new equation, of like degree, whose roots will be less by *r*, than the given equation's.

2. Again, multiply *A* by *r*, and add the product to the last number under *B;* multiply the sum by *r*, and add the product to the last number under *C;* and so on, to the coefficient of *x;* and repeat the like operations, stopping each succeeding time at one coefficient *farther and farther on the left*, until the last sum falls under *B*.

The coefficient *A*, and the last sums under *B*, *C*, &c., will be the *coefficients* of y^n, y^{n-1}, y^{n-2}, &c., in the new equation.

3. Find the *first decimal* figure in the required root, by dividing the *second member* of the new equation by the *coefficient* of *y*.

This first decimal figure will sometimes be incorrect, and should therefore always be tested, by substituting it, and also the one which is next greater or less, for *y*, in the equation (290).

4. Find the *second decimal* figure in the required root, from the *first*, and the new equation, in the same manner in which the first was obtained from *r*, and the given equation; and so proceed for the *third* and following decimal figures in the required root.

The quotient figures found from the successive divisions, become more and more exact as the operation proceeds; so that in the *third* division, the quotient will generally be the true figure required.

When three or four decimal figures have been found as above; several more may usually be determined, by merely *continuing the last division*.

A *negative root* may be found by changing the *alternate signs* of the given equation, and then proceeding as above (282).

EXAMPLE I.

To find a positive root of the Equation

$$(1) \qquad x^3-2x=5.$$

The second power of x is wanting, and we must therefore take 0 for the coefficient of that power.

A	B	C	N
1	+0′	−2′	=5′ (2.0945′
	2	4	4
	2	2	1″
	2	8	.949329
	4	10″	.050671‴
	2	.5481	
	6″	10.5481	
	.09	.5562	
	6.09	11.1043‴	
	.09		10) 1.0 (.1
	6.18		
	.09	11.1043)	.050671 (.0045
	6.27‴		

Ans. $x=2.0945'$

It is readily ascertained that the equation has a root between 2 and 3 (291); and the *integral* part of the root is therefore 2.

Proceeding with the operations according to the Rule, we find that the transformed, or new equation, whose roots are less than those of the given equation by 2, is

$$(2) \qquad y^3+6y^2+10y=1.$$

The value of y, in this equation, is the *decimal* part of the required root; and the highest powers of a *fraction* being small in comparison with the fraction itself, we omit the terms in y^3, and y^2, and take the Equation $10y=1$, from which to find, nearly, at least, the first figure in the value of y; thus:

$$y = 1 \div 10 = .1.$$

But this quotient, .1, needs to be tested in the equation. By substituting .1, and also .09 for y, we shall find that the results will have *contrary signs;* hence, the value of y, is between .09 and .1, and .09 is therefore the first figure in said value.

Proceeding with the *second equation*, and the decimal, .09, in the same manner as with the given equation, and the integer 2, we find the *third* equation

$$(3) \qquad z^3 + 6.27z^2 + 11.1043z = .050671;$$

in which the values of z, are less by .09, than the values of y, in the second equation.

Omitting the terms in z^3, and z^2, we find the *first* figure in the value of z, by dividing the second member by the coefficient of z; thus

$$050671 \div 11.1043 = .004.$$

The next figure 5 in the decimal part of the required root, might be found from the *third*, in the same manner in which that was found from the second; but the 5 will be found at once, by merely continuing the last division.

EXAMPLE II.

To find a positive root of the Equation

$$(1) \qquad x^3 - 17x^2 + 54x = 350.$$

We should find, by trial, that this equation has a root between 14 and 15; but the substitution of numbers containing two or more significant figures, is rather tedious—to avoid which, we may begin with the number of *tens*, in the integral part of the root.

The equation will then be readily found to have a root between 10 and 20; and may be transformed into another, in which the roots shall be respectively diminished by 10; the number 10, being thus made to correspond to r, in the preceding Rule.

$$\begin{array}{rrl}
-17' & +54' & =350'\ (\ 10 \\
\underline{10} & \underline{-70} & \underline{-160} \\
-7 & -16 & 510'' \\
\underline{10} & \underline{30} & \\
3 & 14'' & \\
\underline{10} & & \\
13'' & &
\end{array}$$

$r=10.$

The transformed Equation is thus found to be

$$(2) \qquad y^3+13y^2+14y=510.$$

This equation is readily found to have a root between 4 and 5, and is to be transformed into another, in which the roots shall be diminished by 4.

$$\begin{array}{rrl}
+13'' & +14'' & =510''\ (\ 4. \\
\underline{4} & \underline{68} & \underline{328} \\
17 & 82 & 182''' \\
\underline{4} & \underline{84} & \\
21 & 166''' & \\
\underline{4} & & \\
25''' & &
\end{array}$$

$r=4.$

The roots of the original equation have thus been diminished by the *integral* part, $10+4=14$, and the *decimal* part of the required root is the value of z, in the Equation

$$(3) \qquad z^3+25z^2+166z=182.$$

Omitting the terms in z^3 and z^2, we shall find the first figure in the value of z, nearly, at least, to be

$$182 \div 166 = 1.$$

This figure, however, cannot be the integer 1, and we therefore take the nearest decimal value, .9, which may be tested by substituting it, and also the 1, for z, in the equation.

Other decimal figures may now be found by continuing the operation, as in Example I.

Ans. $x=14.959'$

EXAMPLE III.

To find a value of x, in the Equation

$$x^3=95443993,$$

or, to *extract the cube root* of the second member.

The coefficients of x^2 and x, must be taken at 0.

$$\begin{array}{llll}
1 & +0' & +0^{1\prime} & =95.`443`993' \;(\; 4.57. \\
 & \underline{4} & \underline{16} & \underline{64} \\
 & 4 & 16 & 31.443'' \\
 & \underline{4} & \underline{32} & \underline{27.125} \\
 & 8 & 48'' & 4.318993''' \\
 & \underline{4} & \underline{6.25} & \underline{4.318993} \\
 & 12'' & 54.25 & 0.000000'''' \\
 & \underline{.5} & \underline{6.5} & \\
 & 12.5 & 60.75^{\prime\prime 1} & \\
 & \underline{.5} & \underline{.9499} & \\
 & 13.0 & 61.6999 & \\
 & \underline{.5} & & \\
 & 13.5''' & & \\
 & \underline{.07} & & r=4 \\
 & 13.57 & & \textit{Ans. } x=457.
\end{array}$$

We separate the second member into *periods* of three figures, from right to left, and thus find that the required root will contain *three integral figures* (223).

The greatest *integral cube root* of the left hand period 95 is 4, and it is evident that the two remaining figures of the root will be found in the same manner as if the 4 were the only *integral* figure in the root, and the other two were *decimals.*

We may therefore, in the operation, regard the two right hand *periods* as decimals, if we afterwards assign an integral value to the resulting .57 in the root.

The fourth dividend becoming 0, shows that the *exact* cube root of the second member has been found.

☞By a like method *any of the higher roots of numbers may be extracted.*

On the preceding Rule it needs only to be remarked, by way of further elucidation—

1. When several decimal figures are required in the root, *the operation may be abbreviated* by using in the several *products*, towards the close of the work, only so many decimals as will make the number of decimals in the *dividends* equal to the number required in the root.

2. As the operation is for finding a *positive* root, each dividend and its divisor must have the *same sign*. When, therefore, these terms result with contrary signs, *division will be precluded*, and the next figure of the root must be found by *trial*.

EXERCISES

On Horner's Method of Solving Equations.

1. Find an approximate value of x in the Equation

$$x^3-6x^2+18x=22.$$

Ans. $x=2.3274'$.

2. Find an approximate value of x in the Equation

$$2x^3+4x^2-16x=48.$$

Ans. $x=3.0936'$

3. Find an approximate value of x in the Equation

$$x^4-12x^2+12x=3.$$

Ans. $x=2.858'$.

4. Find an approximate value of x in the Equation

$$3x^5+4x^3-5x=140.$$

Ans. $x=2.0726^2$

5. Find an approximate value of x in the Equation

$$x^5=701583371424,—$$

which is equivalent to *extracting the fifth root* of the second member.

Ans. $x=234$.

Recapitulatory Remarks on the Higher Equations.

(299.) The principles and processes which are to be regarded as mainly useful in reference to the Higher Equations, may be here briefly adverted to, as follows:—

1. A *Root* of an equation is a value of the unknown quantity in the equation; and the number of roots is always equal to the *exponent* of the highest power of the unknown quantity in the equation (255).

2. Two or more of the Roots of an Equation may be *equal;* and an even number of them may be *irrational* or *imaginary* (292).

3. By *Sturm's* Theorem we may determine the number of *real* roots, and thence the number of *imaginary* roots of any given equation. Among the real roots we may, by the same Theorem, also determine the number that are *positive* or *negative* (295).

4. *Integral* roots, either positive or negative, may be found by means of *trial* binomial *divisors* of the equation (268).

5. An *approximate* Root may be found by means of two *trial* numbers, to be substituted for the unknown quantity in the equation, and proportional *corrections* to be found, and applied to those numbers (269).

6. The roots of the Higher Equations may also be determined by *Horner's* Method; and by the same method the *higher roots* of numbers may be extracted (298).

7. When one root of an equation has been determined, the equation may be reduced *one degree lower* by division (253), and the remaining roots may then be found from the *new equation.*

CHAPTER XIII.

CONTINUED FRACTIONS.—APPROXIMATING FRACTIONS.

CONTINUED FRACTIONS.

(300.) A CONTINUED FRACTION is a development of the value of an irreducible proper fraction, or *ratio*,—and arises from dividing both the given terms by the given numerator, proceeding in like manner with the fraction in the resulting *denominator*, and so on; and connecting each succeeding simple fraction, of which the numerator will always be *unity*, by the sign + to the preceding *denominator*.

Continued Fractions are chiefly useful in finding, in lower terms, successive *approximations* to the value of a given fraction, or ratio, whose terms are large, and *prime to each other*.

EXAMPLE.

To find approximate values of the fraction, or ratio,

$$\frac{131}{418}$$

By dividing both the given terms by the numerator, and performing the same operation on the fraction in the resulting *denominator*, we shall find

$$\frac{131}{418}=\frac{1}{3\frac{25}{131}}=\cfrac{1}{3+\cfrac{1}{5\frac{6}{25}}}=\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{4+\cfrac{1}{6}}}}.$$

The *first approximation* to the value of the given fraction, is

$$\frac{1}{3};$$

which is *greater* than the given fraction, because the denominator 3 is less than the true denominator $3\frac{25}{131}$ (147).

The *second approximation* is the continued fraction

$$\frac{1}{3+\frac{1}{5}}=1\div\frac{16}{5}=\frac{5}{16};$$

which is *less* than the given fraction, because, by rejecting the $\frac{6}{25}$, the term $\frac{1}{5}$ becomes *greater* than the true value to be added to the denominator 3 (147), and the result $\frac{5}{16}$, is thus rendered *less* than the given fraction.

The *third approximation* is the continued fraction

$$\frac{1}{3+\frac{1}{5+\frac{1}{4}}}=\frac{1}{3+1\div\frac{21}{4}}=\frac{1}{3+\frac{4}{21}}=\frac{21}{67};$$

which is *greater* than the given fraction, because, by rejecting the $\frac{1}{6}$, the $\frac{1}{4}$ becomes too much to add to the denominator 5, and thence results too small a value to add to the denominator 3, and the value $\frac{21}{67}$, is thus rendered greater than the given fraction (147).

We have thus obtained *three approximations*, the first of which is *greater*, the second *less*, and the third *greater*, than the given fraction

$$\frac{131}{418}.$$

If the remaining term, $\frac{1}{6}$, were included in the computation, the result would be the given fraction.

The simple fractions, whose numerators are *unity*, in a Continued Fraction, may be called *integrant fractions;* each succeeding one of which is to be added to the *denominator* of the preceding one.

In the example just given, the *integrant fractions* are

$$\frac{1}{3},\ \frac{1}{5},\ \frac{1}{4},\ \text{and}\ \frac{1}{6};$$

the first of which is the first approximate value of the fraction developed; the second is to be added to the *denominator* of the first; the third to the denominator of the second, and so on.

Representing the denominators of the successive *integrants* by a, b, c, &c., the successive *approximations* are

$$\text{1st,}\quad \frac{1}{a}; \qquad \text{2d,}\quad \frac{1}{a+\frac{1}{b}}=\frac{b}{ab+1};$$

$$\text{3d,}\quad \frac{1}{a+\frac{1}{b+\frac{1}{c}}} = \frac{bc+1}{c(ab+1)+a};$$

Whatever be the number of *integrants* in the development, the same method of reasoning as in the preceding Example, will show, that—

(301.) The successive approximations are alternately *greater* and *less* than the fraction developed, that is, the whole Continued Fraction.

With these preliminaries we proceed to the theory of

Approximating Fractions.

(302.) The successive *approximations* to the value of a fraction, or *ratio*, which has been developed into a Continued Fraction (300), may be readily ascertained, as follows:—

I. The *first approximation* is the same as the first *integrant* fraction in the Continued Fraction.

II. The *second approximation* will have for its numerator the *denominator* of the second integrant; and for its denominator the product of the denominators of the first and second integrants $+1$.

III. The two terms of the *third approximation* will be found by multiplying the *denominator* of the third integrant into the corresponding terms of the second approximation, and adding the corresponding terms of the first approximation.

IV. Each *succeeding approximation* may be found from the *two which next precede it*, in the same manner in which the third is obtained from the second and first.

The first three of these methods are easily deduced from the Example before given. To demonstrate the fourth, let the first three approximations be denoted by

$$\frac{N'}{D'}, \quad \frac{N''}{D''}, \quad \frac{N'''}{D'''}.$$

From the manner in which the third approximation is derived from the second and first, as before stated, we shall then have

$$N'''=cN''+N';$$
$$D'''=cD''+D'.$$

Now it is evident that the third approximation passes into the *fourth*, when the *fourth integrant* $\frac{1}{d}$ is added to the denominator c of the third integrant; hence

$$N^{\text{iv}}=(c+\frac{1}{d})\ N''+N';$$

$$D^{\text{iv}}=(c+\frac{1}{d})\ D''+D'.$$

The second members of these last equations are equal to

$$\frac{cdN''+N''+dN'}{d},$$
$$\text{and}\ \frac{cdD''+D''+dD'}{d}.$$

In the numerators of these fractions, we have, by decomposing the terms containing d, and recurring to the first and second equations.

$$cdN''+dN'=d(cN''+N')=dN''';$$
$$cdD''+dD'=d(cD''+D')=dD'''.$$

Hence, by substituting these values in the numerators of the preceding fractions, and omitting the common denominator d (81), we find

$$\frac{N^{\text{iv}}}{D^{\text{iv}}}=\frac{dN'''+N''}{dD'''+D''}.$$

This shows that the fourth approximation may be found from the third and second, in the same manner in which the third is obtained from the second and first; and it is evident that a similar demonstration is applicable to the fifth and subsequent approximations.

(**303.**) Any one of the approximating fractions differs in value from the fraction developed, by less than a *unit* ÷ the product of the denominators of that and the succeeding approximation.

To establish this proposition, it must first be shown, that

If any two consecutive approximations be reduced to a *common denominator*, the difference of the resulting numerators will be a *unit*.

This is readily shown for the first and second approximations: thus, from the Example already given, we have

$$\frac{1}{a}-\frac{b}{ab+1}=\frac{ab+1-ab}{a(ab+1)}=\frac{1}{a(ab+1)}.$$

For a general demonstration, let the first three approximations be represented by

$$\frac{N'}{D'},\quad \frac{N''}{D''},\quad \frac{N'''}{D'''}.$$

By reducing to a common denominator, and subtracting, we have

$$\frac{N'}{D'}-\frac{N''}{D''}=\frac{N'D''-N''D'}{D'D''};$$

$$\frac{N''}{D''}-\frac{N'''}{D'''}=\frac{N''D'''-N'''D''}{D''D'''}.$$

But we have already seen that

$$N'''=cN''+N', \text{ and } D'''=cD''+D' \quad (302\ldots\text{III});$$

and by substituting these values of N''' and D''' in the numerator of the second member of the second equation, and simplifying, we have

$$\frac{N''}{D''}-\frac{N'''}{D'''}=\frac{N''D'-N'D''}{D''D'''}.$$

The numerator of $N''D'-N'D''$, of this *second* difference is the same as the numerator $N'D''-N''D'$, of the *first* difference, with the *signs changed*.

Now, the numerator of the first difference has already been found to be $ab+1-ab=+1$; hence, the numerator of the second difference is equal to -1; and by proceeding in the same manner with the *second third*, and *fourth* approximations, we shall find the successive differences to be alternately $+1$ and -1.

The difference between any two consecutive approximations, is therefore equal to a *unit* ÷ by their common denominator, that is, the product of their denominators; and since the value of the fraction developed is *greater* than the one, and *less* than the other, of any two consecutive approximations (301), it differs from either of these less than they differ from each other.

These considerations establish the foregoing proposition. It will hence follow, that—

(304.) Any particular approximation differs in value from the fraction, or ratio, developed, by less than a *unit* ÷ the *square of the denominator* of that approximation.

For the denominators successively increase, and the square of any denominator is therefore less than the product of that denominator × the next denominator.

EXERCISES

On Continued and Approximating Fractions.

1. Find approximate values of $\frac{163}{1493}$. *Ans.* $\frac{1}{9}, \frac{6}{55}, \frac{19}{174}$.

2. Find approximate values of $\frac{329}{1000}$. *Ans.* $\frac{1}{3}, \frac{25}{76}, \frac{76}{231}$.

3. Find approximate values of $2\frac{17}{100}$. *Ans.* $2\frac{1}{5}, 2\frac{1}{6}, 2\frac{8}{47}$.

4. Find approximate values of 3.14159, which is the ratio, nearly, of the circumference to the diameter of a circle.

Ans. $3\frac{1}{7}, 3\frac{15}{106}, 3\frac{16}{113}$.

. The pyramid of Cheops in Egypt, which is the loftiest structure ever raised by man, is 543 feet; and the altitude of the loftiest mountain is 29,000 feet. What are approximate ratios of these altitudes?

Ans. $\frac{1}{53}, \frac{2}{107}, \frac{5}{267}$.

6. Find approximate ratios of the altitude of the loftiest mountain, 29000 feet, to the semi-diameter of Earth, taken at 4000 miles.

Ans. $\frac{1}{728}, \frac{3}{2185}, \frac{4}{2913}$.

CHAPTER XIV.

LOGARITHMS AND THEIR APPLICATIONS.

LOGARITHMS.

(305.) LOGARITHMS of numbers are the *exponents* of the *powers* to which some assumed number, called the *base* of the logarithms, must be raised, to produce the given numbers.

Thus, in the Equations

$$a^x = n \text{ and } a^y = m,$$

a may represent the constant *base*, and then the exponents x and y will represent the *logarithms* of the numbers, n and m, respectively.

Unity could not be taken for the *base* of a system of logarithms, since the series of numbers could not be obtained from the *powers of unity* (199). Any other number might be used for this purpose, and hence there may be—

Different Systems of Logarithms.

(306.) The *Common System* of Logarithms, that is, the system in common use, is *based* on the number 10; all numbers being regarded, in this system, as *powers of* 10.

Commencing with the 0 power of the *base* 10, we have

$(10)^0 = 1$; $(10)^1 = 10$; $(10)^2 = 100$; $(10)^3 = 1000$; $(10)^4 = 10000$, &c.; the *exponents*, 0, 1, 2, 3, 4, &c., being, respectively, the *logarithms* of the numbers 1, 10, 100, 1000, 10000, &c.

Common Logarithms, also called *Briggs's* Logarithms, from the name of their author, are used to facilitate tedious operations in the multiplication, division, involution, and evolution of numbers.

Another System of Logarithms is that of *Napier*, the inventor of logarithms. The base of this system will be shown hereafter.

Naperian Logarithms are employed in some of the applications of the *Differential and Integral Calculus*: they also facilitate the computation of common logarithms.

GENERAL PROPERTIES OF LOGARITHMS.

The General Theory of Logarithms is contained in the following propositions, on which the methods of logarithmic calculation are mainly dependent.

Logarithm of Unity, &c.

(307.) The logarithm of *unity* is always 0, and the logarithm of the *base* of logarithms is always *unity*.

For whatever be the value of the base a, we have

$$a^0=1(48);\ \text{and}\ a^1=a.$$

In the first equation, 0 is the logarithm of 1, and in the second, 1 is the logarithm of a.

Logarithm of a Product.

(308.) The Logarithm of the *product* of two or more numbers, is equal to the *sum of the logarithms* of those numbers.

In the equations $a^x=n$ and $a^y=m$, the exponents, x and y, are the logarithms of n and m, for the base a; and by multiplying the equations together, we have

$$a^{x+y}=n\ m\ (41),$$

in which $x+y$ is the logarithm of the product $n\ m$.

Logarithm of a Quotient.

(309.) The logarithm of the *quotient* of two numbers, is equal to the logarithm of the dividend *minus* the logarithm of the divisor.

Dividing the Equation $a^x=n$ by the Equation $a^y=m$, we have

$$a^{x-y}=n\div m\ (47),$$

in which $x-y$ is the logarithm of the quotient $n\div m$, for the base a; while x and y are the logarithms of n and m.

Logarithm of a Power or Root.

(310.) The logarithm of any *power* or *root* of a number, is equal to the *exponent* of the power or root × the logarithm of the number.

If any particular power or root, as the 3d, be found of each side of the Equation $a^x = n$, we shall have

$$a^{3x} = n^3 \ (200), \text{ or } a^{\frac{x}{3}} = n^{\frac{1}{3}} \ (208);$$

in which the exponent $3x$, 3 times the logarithm of n, is the logarithm of the power n^3; and $\frac{x}{3}$ is the logarithm of the root $n^{\frac{1}{3}}$.

It will be observed, that multipling the logarithm of the number by the exponent of the root to be extracted, is equivalent to *dividing* the logarithm by the *integer* corresponding to the root.

Logarithm of 0.

(311.) The logarithm of 0 is *infinite;* and is *negative* or positive, according as the *base* of the system of logarithms is *greater* or less than *unity.*

First; let the base a be *greater* than unity, and let

$$\frac{1}{a^x} = a^{-x} = 0 \ (77),$$

in which the exponent $-x$ is the logarithm of 0.

That the value of the fraction in the first member may be *infinitely small,* that is, $=0$, the denominator a^x must be *infinitely great* (147); but no *finite power* of the finite quantity a can be infinite; hence the value of the exponent x is *infinite.*

Secondly; let the base a be *less* than unity; and let

$$a^x = 0.$$

The value of a being a *proper fraction,* the higher powers of a will continually *diminish;* but no *finite* power of a can be *infinitely small,* that is, $=0$; hence the value of x is *infinite.*

The logarithm of 0 is therefore *infinite,* and is *negative,* when the *base* is greater than *unity,* but positive when the base is less than unity.

Progression of Logarithms.

(312.) If a series of numbers be taken in Geometrical Progression, their *logarithms* will be in Arithmetical Progression.

Let the geometrical progression be represented by

$$b,\ br,\ br^2,\ br^3,\ br^4,\ \&c.$$

Let a be the *base* of a system of logarithms, x the logarithm of b, and y the logarithm of r; then

$$a^x = b, \text{ and } a^y = r;$$

and by multiplying the corresponding members of these equations into each other continually, we shall have

$$a^x = b;\ a^{x+y} = br;\ a^{x+2y} = br^2;\ a^{x+3y} = br^3,\ \&c.$$

In these equations the exponents are the logarithms of the terms in the assumed *geometrical* series, and these logarithms form an *arithmetical* progression whose first term is x and common difference y.

COMPUTATION OF LOGARITHMS.

(313.) From the definition of Logarithms, we have the series

0, 1, 2, 3, 4, 5, 6, &c.,

for the logarithms of the *corresponding powers* of whatever number is assumed for the *base* of the system of logarithms.

Thus, in the common system, whose base is 10,

the logarithm of 1	or 10^0	is 0;
the logarithm of 10	or 10^1	is 1;
the logarithm of 100	or 10^2	is 2;
the logarithm of 1000	or 10^3	is 3;
the logarithm of 10000	or 10^4	is 4;

The Logarithm of a number which is not equal to any *integral power* of the base, as 5 in the common system, may be found by series of approximating geometrical and arithmetical *means.*

For example, to compute the *common logarithm* of 5.

We have seen that the logarithms of 1 and 10 are 0 and 1, respectively; and, by a property already demonstrated, a geometrical *mean* between any two numbers will have for its *logarithm* the arithmetical *mean* between the logarithms of those numbers (312).

Hence, the log. of $(1\times10)^{\frac{1}{2}}$ or of 3.162277' (189),
is $(0+1)\div2$ or .5 (179);

the log. of $(3.162277\times10)^{\frac{1}{2}}$ or of 5.623413',
is $(.5+1)\div2$ or .75;

the log. of $(3.162277\times5.623413)^{\frac{1}{2}}$ or of 4.216964,
is $(.5+.75)\div2$ or .625;
&c. &c.

The computation proceeds by finding successive geometrical means; taking for the *extremes*, in each case, the numbers which shall be found next less and greater than 5; and finding also the corresponding arithmetical means or *logarithms*.

The 22d geometrical mean would be found to be 5.00000', and the corresponding arithmetical mean or *logarithm* .698970'.

In like manner, the logarithm of any other number not included in the series of numbers whose logarithms are known, might be computed.

The *Logarithm of a number is readily obtained* when that number is the *product* or quotient of two others whose logarithms are known (308 & 309).

Thus, the logarithm of 2 or $\frac{10}{5}$ is $1-.69897=.3103$.

We may also very readily find the logarithm of a number which is any *power* or root of another whose logarithm is known (310).

Thus, from the logarithm of 2 we shall obtain the logarithms of 4, 8, 16, &c., by multiplying by 2, 3, 4, &c.

By the preceding method, the logarithms of the *prime* numbers are said to have been at first computed. More expeditious methods are now known, the most common of which is by means of—

Logarithmic Series.

(314.) The Logarithm of a number may be developed in a *series of terms* depending on the given number and the *base* of the logarithm, and a Formula be thence derived for the ready computation of logarithms.

Let $a^x = N$; in which x is the logarithm of N;

assume $a = 1 + c$, and $N = 1 + d$;

then $(1+c)^x = 1 + d$;

and $(1+c)^{xy} = (1+d)^y$, for every value of y.

Developing each side of the last equation according to the Binomial Theorem—observing that *unity* with any exponent whatever is *unity*, we find

$$1 + xyc + \frac{xy(xy-1)}{2}c^2 + \frac{xy(xy-1)(xy-2)}{2\times3}c^3 + \&c.$$
$$= 1 + yd + \frac{y(y-1)}{2}d^2 + \frac{y(y-1)(y-2)}{2\times3}d^3 + \&c.$$

Canceling 1 from both sides of this Equation, dividing by y, and decomposing the first member, we have

$$x\left(c + \frac{xy-1}{2}c^2 + \frac{(xy-1)(xy-2)}{2\times3}c^3 + \&c.\right)$$
$$= d + \frac{y-1}{2}d^2 + \frac{(y-1)(y-2)}{2\times3}d^3 + \&c.$$

As y may have any value whatever in the equation in which it is first employed, we will now suppose that $y = 0$. The last Equation will thus become

$$x(c - \frac{c^2}{2} + \frac{c^3}{3} - \&c.) = d - \frac{d^2}{2} + \frac{d^3}{3} - \&c.$$

From this Equation, we find

$$x = \text{ log. of } N, \text{ or of } (1+d) = \frac{d - \frac{1}{2}d^2 + \frac{1}{3}d^3 - \&c.}{c - \frac{1}{2}c^2 + \frac{1}{3}c^3 - \&c.}$$

And by substituting for d and c their values, $N-1$ and $a-1$, from the second and third equations, there results

$$\log.\text{ of } N=\frac{(N-1)-\frac{1}{2}(N-1)^2+\frac{1}{3}(N-1)^3-\&c.}{(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\&c.}$$

In this Equation we have the logarithm of a number, N, expressed in terms of the number itself and the base, a, of the logarithm; the numerator of the expression being a function of the number, N, and the denominator a function of the base, a. It is necessary, however, to reduce the expression to a form which shall be better adapted to numerical computation.

The *reciprocal* of the denominator $(a-1)-\frac{1}{2}(a-1)^2$, &c., will be a *multiplier* of the numerator in the value of the logarithm of N.

This multiplier is called the *modulus* of the system of logarithms, and is commonly represented by M.

From the last equation but one, we shall then have, according as d is *positive* or *negative*,

$$\log.\text{ of }(1+d)=M(d-\tfrac{1}{2}d^2+\tfrac{1}{3}d^3-\tfrac{1}{4}d^4+\&c.);\text{ or}$$
$$\log.\text{ of }(1-d)=M(-d-\tfrac{1}{2}d^2-\tfrac{1}{3}d^3-\tfrac{1}{4}d^4-\&c.);$$

and by subtracting this Equation from the former,

$$\log.(1+d)-\log.(1-d)=2M(d+\tfrac{1}{3}d^3+\tfrac{1}{5}d^5+\&c.);$$
$$\text{or, }\log.\overline{(1+d)\div(1-d)}=2M(d+\tfrac{1}{3}d^3+\tfrac{1}{5}d^5+\&c.)\ (309).$$

These Equations being true for every value of d, we may assume

$$d=\frac{1}{2n+1};$$

$$\text{then }1+d=\frac{2n+2}{2n+1};\ 1-d=\frac{2n}{2n+1};\ \text{and }(1+d)\div(1-d)=\frac{n+1}{n};$$

and, by substituting these values in the last logarithmic equation, we have, considering that the logarithm of a *quotient* is equal to the logarithm of the dividend *minus* the logarithm of the divisor,

$$\log.(n+1)-\log.n=2M\left(\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\&c.;\right)\text{ or}$$

$$\log(n+1)=\log.n+2M\left(\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\&c.\right)$$

This last equation furnishes a *Formula* by which the logarithm of the number $(n+1)$, may be computed, when the logarithm of n and the value of the *modulus*, M, are known.

The logarithm of *unity* being always 0 (307...I), we may compute in succession the logarithms of 2, 3, &c. The value of M, for the common system, will be shown, after an explanation of

NAPERIAN LOGARITHMS.

(315.) In the *Naperian* System of Logarithms, the *modulus* M, in the logarithmic formula, is taken equal to *unity*.

The Formula for computing these logarithms is therefore

$$\text{Log.}'(n+1)=\text{log.}'n+2\left(\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\text{\&c.};\right)$$

an accent $'$ being affixed to distinguish these from common logarithms.

We shall present a few examples of the computation of *Naperian* logarithms, and then determine the *modulus* for the common system.

Making n successively equal to 1, 2, 3, &c., and recollecting that the logarithm of 1 is always 0; the last Formula gives,

$$\text{log.}'\,2=\text{log.}'\,1+2\left(\tfrac{1}{3}+\frac{1}{3.3^3}+\frac{1}{5.3^5}+\frac{1}{7.3^7}+\cdot\cdot\right)=0.693147$$

$$\text{log.}'\,3=\text{log.}'\,2+2\left(\tfrac{1}{5}+\frac{1}{3.5^3}+\frac{1}{5.5^5}+\frac{1}{7.5^7}+\cdot\cdot\right)=1.098612$$

$$\text{log.}'\,4=\text{log.}'\,2+\text{log.}'\,2,\ \text{or}\ 2.\ \text{log.}'\,2\ (308),\ =1.386294$$

$$\text{log.}'\,5=\text{log.}'\,4+2\left(\tfrac{1}{9}+\frac{1}{3.9^3}+\frac{1}{5.9^5}+\frac{1}{7.9^7}+\cdot\cdot\right)=1.609438$$

&c. &c.

$$\text{log.}'\,10=\text{log.}'5+\text{log.}'\,2\ \ (308),\qquad =2.302585$$

&c. &c.

Naperian Logarithms are sometimes called *hyperbolic* logarithms, from a relation which they bear to the curve of an *hyperbola;* and also *natural* logarithms, from the *modulus* of the system being *unity*.

We are now prepared to find the

Modulus of the Common System.

(316.) The *modulus* of the common system of logarithms, is the multiplier which reduces the Naperian logarithm of a number to the *common* logarithm of the same number; and is always equal to the *reciprocal* of the Naperian logarithm of the *base* 10, of the common system.

For, let $10^x = N$, in which x represents the common logarithm of N; then by taking the Naperian logarithm of each side of the equation, we have

$$x \log.' 10 = \log.' N \ (310);$$

$$\text{which gives } x = \log.' N \div \log.' 10 = \log.' N \times \frac{1}{\log.' 10}.$$

The multiplier $\frac{1}{\log.' 10}$ is the modulus which, multiplied in to the Naperian logarithm of the number N, produces the common logarithm x, of the same number.

We have already found $\log.' 10 = 2.302585$; hence the modulus M, of the common system of logarithms, is $1 \div 2.302585 = .434294$.

The *Common Logarithm of a number will therefore be found* by multiplying the Naperian logarithm of the number by .434294.

Since the Logarithm of a number in the common, or in any system, is found by multiplying the Naperian logarithm of the number by the *modulus* of the other system, it follows that

(317.) The Logarithms of the same number in *different systems*, are to each other as the *moduli* of the respective systems.

Naperian Base of Logarithms.

(318.) The *base* of the Naperian system of logarithms, is that number whose common logarithm is equal to the *modulus* .434294 of the common system.

For, let the Naperian *base* be represented n; then the Naperian logarithm of n being 1 (307); and the *modulus* of this system being also 1 (315); we have

$$\text{com. log. } n : 1 : : .434294 : 1 \ (317),$$

which gives the com. log. $n = .434294$.

From a Table of common logarithms, to be presently explained, the number whose common logarithm is .434294, is found to be 2.718282 which is therefore the Naperian *base*

There will be no further occasion to refer to Naperian logarithms in the present work, and what follows will therefore be understood of *common* logarithms; this system being usually designated by the simple term logarithms.

Characteristics of Logarithms.

(319.) The *Characteristic* of a logarithm is the *integral* part of it; and is always equal to the number of places by which the highest order of units in the corresponding *natural* number is removed from *simple units.*

Thus, the logarithms of 100 and 1000 being 2 and 3, respectively (313), the logarithm of any number between 100 and 1000, as of 475, will be 2 + a *fraction;* the *characteristic* 2, being equal to the number of places by which the 4 *hundreds* is removed from *units.*

Logarithms of Decimal Fractions.

(320.) The logarithm of a Decimal Fraction has the same decimal part as if the given fraction were an *integer;* but its characteristic is *negative*, and equal to the number of places by which the first significant figure in the given decimal, is removed from *units.*

Thus, the logarithm of 5 being .69897, as already found; the logarithm of 1000 being 3; and the logarithm of a *quotient* being equal to the logarithm of the dividend *minus* that of the divisor (309), we have

$$\text{the log. of } .005, \text{ or of } \frac{5}{1000}, = .69897 - 3 = \bar{3}.69897$$

the characteristic $\bar{3}$, *negative* 3, being equal to the number of places by which the 5 in the given decimal is removed from the place of *units.*

The sign −, is set over the 3 to show that only this characteristic is *negative;* the decimal part of a logarithm being always *positive.*

The first statement respecting the numerical value of the *characteristic* (319), evidently includes the second (320); but the following may be found a more convenient guide in practice.

(321.) The *characteristic* of the Logarithm of any number greater than *unity*, is *one less* than the number of *integral* figures in the given number; but

The *characteristic* of the Logarithm of a *decimal fraction* is *negative*, and *one more* than the number of 0's between the decimal point and the first significant figure in the given fraction.

(322.) The *decimal part* of the logarithm of any number, is the same as that of the product, or quotient, of the number multiplied, or divided, by 10, or 100, or 1000, &c.

This is evident, from considering, that as the logarithms of 10, 100, 1000, &c., are 1, 2, 3, &c., the only effect of the multiplication, or division, would be to increase, or diminish, the *characteristic* of the logarithm of the given number; the decimal part of the logarithm remaining the same (308), (309).

TABLE OF LOGARITHMS.

(323.) The TABLE appended to this work contains the computed logarithms of all numbers from 1 to 10,000; or rather the *decimal* parts of the logarithms, since the *characteristics*, being always known (319), are omitted for numbers above 100.

These logarithms are given to *six places of decimals*, which are sufficient for all ordinary purposes.

On the first page of the Table, in the columns marked *N* at the top, are the *natural* numbers from 1 to 100, with the logarithm of each number immediately on the right, in the next column.

On the second and following pages of the Table, in column *N*, are the *natural* numbers from 100, to 999, with the *decimal* part of the logarithm of each immediately on the right, in the next column—*vacant places* in the latter requiring to be supplied with the *two initial* figures which stand next above the vacant place.

Thus, the decimal part of the logarithm of 138, is .139879.

With the proper *characteristic* prefixed, this logarithm becomes

2.139879 (321).

When the natural number consists of *four figures*, its three left hand figures must be found in column *N*, and the other at the head of one of the other columns. The *decimal* part of the logarithm will be the four figures in the latter column, which stand opposite to the said three left hand figures, and the *two initial* figures from the column under 0.

Thus, the decimal part of the logarithm of 2035, is .308564.

With the proper *characteristic* prefixed, this logarithm becomes

3.308564 (321).

Points, or *dots*, . . , occurring among the logarithms in the Table, denote that such places must be supplied with 0's, and that from thence the *two initial* figures from column 0 must be taken from the line next *below*.

Thus, the decimal part of the logarithm of 2045, is .310693;
and the decimal part of the logarithm of 2048, is .311330.

The numbers in the column D, on the right of the Table, are the *differences* between consecutive logarithms on the same horizontal line; and are thus arranged to facilitate the calculation when we wish

(324.) *To find from the Table the Logarithm of a Number containing more than four figures.*

1. Find the decimal part of the logarithm of the four left hand figures of the given number, in the manner explained above.

2. Multiply the corresponding number in column D by the remaining figures of the given number, and *reject from the right of the product* as many figures as there are in the multiplier.

3. Add the other part of the product to the logarithm already found—observing to increase the right hand figure by a *unit*, when the *rejected* figures, regarded as decimals, are 5 *tenths* or more.

4. The result obtained will be the *decimal part* of the required logarithm, to which the proper *characteristic* must be prefixed (321).

EXAMPLE.

To find the logarithm of 176.421.

The decimal part of the logarithm of 1764, is .246499.

On the same horizontal line, in column D, we find 246, which, multiplied by the remaining figures 21 of the given number, gives the product 5166.

Rejecting two figures from the right of this product—there being two figures in the multiplier 21—we have 52 to add to the logarithm already found.

The decimal part of the required logarithm is therefore

$$246499+52=.246551;$$

and the characteristic, or *integral* part, is 2 (321); so that

The log. of 176.421, is 2.246551.

The preceding method of finding the logarithm of a number containing more than *four figures*, proceeds upon the supposition that

The differences of numbers are proportional to the differences of their logarithms—which is very nearly true with large numbers, as an examination of the Table will indicate.

In the preceding Example, the number 246, which is in reality .000246, taken from column D, is the *difference of the logarithms* of 1764 and 1765.

If therefore we add .21 of 246 to the decimal part of the logarithm of 1764, we shall obtain the decimal part of the logarithm of 1764.21, which is the same as that of 176.421 (322.)

EXERCISES

In finding Logarithms from the Table.

1. Find the logarithm of 25679. *Ans.* 4.409578.
2. Find the logarithm of 3043.62. *Ans.* 3.483391.
3. Find the logarithm of 189.037. *Ans.* 2.276547.
4. Find the logarithm of 45.128. *Ans.* 1.654446.
5. Find the logarithm of 373934. *Ans.* 5.572794.
6. Find the logarithm of 2.47344. *Ans.* 0.393301.
7. Find the logarithm of .639853. *Ans.* $\bar{1}.806080$.
8. Find the logarithm of .03848. *Ans.* $\bar{2}.585235$.

The logarithm of a vulgar fraction is evidently equal to the logarithm of its numerator *minus* the logarithm of its denominator (309).

9. Find the logarithm of $\frac{25}{379}$. *Ans.* $\bar{2}.819301$.
10. Find the logarithm of $\frac{277}{1260}$. *Ans.* $\bar{1}.342109$.
11. Find the logarithm of $\frac{300}{2763}$. *Ans.* $\bar{1}.035740$.

(325.) *To find from the Table the number corresponding to any given Logarithm.*

1. Find the first *two decimal* figures of the given logarithm in the column under 0, and the other four, if possible, in that or one of the right hand columns; then in column N, opposite to the *latter part* of the logarithm, will be the *three left hand* figures, and at the head of the column containing the latter part, will be the other figure of the required number.—But

2. When the exact decimal part of the logarithm *is not contained in the Table*—find the four figures which correspond to the *next less tabular logarithm;* subtract this logarithm from the given one; divide the remainder by the corresponding number in column D, prefixing a 0 to the quotient for each 0, if any, annexed to the remainder *without finding a significant quotient figure;* and annex the result to the four figures already found.

3. The number obtained by either of these methods, must be adjusted to the *characteristic* of the given logarithm, by *pointing off decimals*, or annexing 0's if necessary (321).

EXAMPLE.

To find the number whose logarithm is 2.491090.

The exact decimal part of this logarithm not being contained in the Table, we take the four figures 3098, which correspond to the next less tabular logarithm 491081.

140) 9000 (064.

Having subtracted the less tabular logarithm from the decimal part of the given logarithm, we divide the remainder 9, by the corresponding number 140 in column D, prefixing a 0 to the quotient 64, because one 0 was annexed to the 9 without obtaining a significant quotient figure; and annex 064 to the four figures 3098, already found in the Table.

We thus obtain the number 3098064, in which three figures must be *integral*, since the characteristic of the given logarithm is 2 (321).

This method is but the converse of the preceding (324), and requires no separate elucidation.

EXERCISES

In finding, from the Table, a number answering to a given Logarithm.

1. Find the Number whose logarithm is 1.241297.
Ans. 17.43.
2. Find the Number whose logarithm is 0.338130.
Ans. 2.17836.
3. Find the Number whose logarithm is 2.753683.
Ans. 567.129.
4. Find the Number whose logarithm is $\bar{1}.867372$.
Ans. .736837.
5. Find the Number whose logarithm is $\bar{3}.723460$.
Ans. .00529004.
6. Find the Number whose logarithm is .003985.
Ans. 1.00921.
7. Find the Number whose logarithm is 4.536029.
Ans. 34358.09.

APPLICATIONS OF LOGARITHMS.

From principles already demonstrated the following Applications of logarithms are immediately deducible.

(326.) *To Multiply two or more Numbers together.*

Add together the *logarithms* of the given numbers, and find the *natural* number corresponding to the *sum* (308).

(327.) *To Divide one Number by another.*

Subtract the logarithm of the *divisor* from the logarithm of the dividend, and find the natural number corresponding to the *difference* (309).

(328). *To raise any Power of a Number.*

Multiply the logarithm of the number by the *exponent* of the power, and find the natural number corresponding to the *product* (310).

(329.) *To extract any Root of a Number.*

Divide the logarithm of the number by the *integer* corresponding to the root, and find the natural number corresponding to the quotient (310).

Arithmetical Complement of Logarithms.

(330.) The *arithmetical complement* of a logarithm is what remains when the logarithm is subtracted from 10; and may be *added* in calculations in which the logarithm itself should be *subtracted*, if 10 be rejected from the sum.

Thus, the arithmetical complement of Log. a, is

$$10-\log. a.$$

By adding this complement to Log. b, we have

$$\log b+10-\log. a;$$

and by rejecting 10 from this sum, we obtain

$$\log. b-\log. a,$$

which is the difference that would be found by *subtracting* the logarithm whose complement was used.

The Arithmetical Complement is readily obtained by subtracting the first right hand figure of the logarithm from 10, and each of the other figures from 9.

The Arithmetical Complement of a logarithm is used to abbreviate a calculation in which *multiplication and division are both concerned.*

EXAMPLE I.

To find, by means of logarithms, a *fourth proportional* to the three numbers, 20, 40, and 60 (150).

$$\begin{array}{rl} \text{Log. of } 60 = & 1.778151. \\ \text{Log. of } 40 = & \underline{1.602060} \\ & 3.380211 \\ \text{Log. of } 20 = & \underline{1.301030} \\ & 2.079181 \end{array}$$

To find the logarithm of the *product* of the second and third terms, we add together the logarithms of those terms; and then to find the logarithm of the *quotient* of this *product* ÷ the first term, we subtract the logarithm of the first term.

To abbreviate the calculation, we take the

$$\begin{array}{rl} \textit{Arith. Comp.} \text{ of Log. of } 20 = & 8.698970. \\ \text{Log. of } 40 = & 1.602060. \\ \text{Log. of } 60 = & \underline{1.778151} \\ & 2.079181 \end{array}$$

The Arithmetical Complement of the logarithm of the first term, which is the *divisor*, and the logarithms of the second and third terms, which are the *factors of the dividend*, are added into one sum, and 10 is rejected from that sum.

The natural number 120, corresponding to the logarithm 2.079181, is the *fourth proportional* required.

☞ When the divisor is the product of *two or more factors*, the arithmetical complement of the logarithm of each of those factors may be employed; and then as many 10's must be rejected from the sum as there are arithmetical complements employed.

EXAMPLE II.

To extract the *fifth root* of .03795.

The logarithm of the given number is $\bar{2}.579212$, which is to be divided by 5, for the logarithm of the required *root* (329).

To find an *exact integral characteristic* for the logarithm sought, the *negative* characteristic $\bar{2}$ must be expressed by its equivalent

$$\bar{2}-3+3=\bar{5}+3\,;$$

then $\bar{5}+3.579212 \div 5 = \bar{1}.715842$; and the number corresponding to this logarithm is .519807, which is the *required root*.

This method of using an *equivalent* characteristic must be adopted, whenever a *negative* characteristic is not a multiple of the number by which it is to be divided.

EXERCISES

In Logarithmic Calculations.

1. Find a *fourth proportional* to the numbers
25, 175, and 105.9. *Ans.* 741.3.

2. Find a fourth proportional to the numbers
375, 4384, and 12.637. *Ans.* 147.734.

3. Find a fourth proportional to the numbers
54 × 13, 5.75, and .34762. *Ans.* .002847.

4. Find a fourth proportional to the numbers
39 × .75, 39436, and 284 × .05. *Ans.* 19144.9.

5. Find a *mean proportional* between the numbers
1.734, and 382.635. *Ans.* 25.7582.

6. Find a mean proportional between the numbers
.2746, and 934893. *Ans.* 506.676.

7. Find a mean proportional between the numbers
34.3×9, and $.733 \times .045$. *Ans.* 3.191.

8. Find the *ratio* of a *geometrical progression* whose first term is 9, number of terms 6, and last term 3087. *Ans.* 3.21408.

9. Find the ratio of a geometrical progression whose first term is 547, number of terms 7, and last term 5470. *Ans.* 1.46779.

10. Find the ratio, and thence the *second term*, of a geometrical progression whose first term is 5, number of terms 6, and last term 1000.
Ans. Ratio, 2.8853 ; second term, 14.4265.

11. Find the value of x in the Equation

$$12^x = 150.75.$$

The first member of this equation forms what is called an *exponential* quantity, by which is meant a quantity which has an *unknown* quantity for its *exponent;* and the equation itself is an *Exponential Equation.*

Taking the logarithm of each member of the equation, we have

$$x \log. 12 = \log. 150.75 \ (310),$$

$$x = \frac{\log. 150.75}{\log. 12}.$$

Ans. $x = 2.0184$.

12. Find the value of x in the Equation

$$x^x = 1000.$$

By taking the logarithm of each side of the Equation, we have

$$x \log. x = \log. 1000 = 3 \ (310).$$

This last Equation may be solved by the method of approximation explained in a former Chapter (269).

Ans. $x = 4.555$.

13. Find the value of x in each of the Equations

$$20^x = 2000, \text{ and } x^x = 2000.$$

Ans $x = 2.537$; 4.827.

COMPOUND INTEREST.

(331.) *Simple Interest* is interest on a given principal only (173).

Compound Interest is interest on both principal and *interest,* when the latter remains unpaid after it has become due.

The interest is usually understood to be compounded *annually;* and the *rate per cent* of interest always means the simple interest of \$100 for *one year.*

Let P represent any given *Principal* at compound interest;
r the *ratio* of percentage, or interest, of \$1 for 1 year (171);
n the number of *years* the interest is accruing;
A the whole *amount* of principal and interest; then

$\$(1+r)$ is the amount of \$1 for *one year;* $P(1+r)$ is the amount of P dollars for *one year;* and since the amount for the 1st year is the *principal* for the 2d, and so on; we have, by multiplying each principal by $(1+r)$,

$P(1+r)$ = the amount for *one* year;
$P(1+r)^2$ = the amount for 2 years;
$P(1+r)^3$ = the amount for 3 years;

and so on, in *geometrical progression,* for any number of years; and therefore

$A=P(1+r)^n$, the amount for n years.

Taking the *Logarithm* of each side of this Equation, we have

(1) $\log. A=\log. P+n \log. (1+r)$, (308); (310);
(2) $\log. P=\log. A-n \log. (1+r)$;
(3) $\log. (1+r)=(\log. A-\log. P)\div n$
(4) $n=(\log. A-\log. P)\div \log.(1+r)$.

If the interest were compounded *semi-annually,* $2n$ would take the place of n, and $\frac{1}{2}r$ would take the place of r; and formula (1) would then become

$$\log. A=\log. P+2n. \log. \left(1+\frac{r}{2}\right).$$

Other modifications of the formulas will adapt them to the particular conditions of the problem to be solved.

Thus, to find the time in which a given principal would *double* itself at Compound Interest, we would substitute $2P$ for A.

EXAMPLE.

To find the *rate per cent* at which $100, or any other principal, would double itself in 10 years, allowing the interest to be compounded annually.

By making the proper substitutions in formula (3), we have

$$\begin{aligned}\log.\,(1+r)&=(\log.\,200-\log.\,100)\div 10,\\&=(2.301030-2)\div 10,\\&=0.301030\div 10,\\&=0.030103.\end{aligned}$$

The number whose logarithm is 0.030103, is $1.0717=1+r$; then

$$r=1.0717-1=.0717,$$

which is the *ratio of percentage*, or interest, of $1 for 1 year.

The required *rate per cent* is therefore $.0717\times 100=7.17$.

Problems in Compound Interest.

1. What would $500 amount to in 20 years, allowing compound interest at 6 per cent? *Ans.* $1603.57.

2. What principal would amount to $50,000 in 50 years, allowing compound interest at 5 per cent? *Ans.* $4360.34.

3. Which would be the most advantageous to the lender, the simple interest of $1000 for 10 years, at 6 per cent, or the compound interest at 5 per cent? *Ans.* The comp. int. by $1028.88.

4. A debt of $3785.75. will be due in 5 years, without interest; what is the *present value* of the debt, allowing money to be worth 6 per cent at compound interest?

The *present* value is that *principal* which, at compound interest, would amount to the debt by the time the debt becomes due. *Ans.* $2828.92.

5. A is indebted to B $3794.25, to be paid in two equal instalments, in 3 and 6 years, without interest. What is the present value of the debt, on the supposition that money will produce 5 per cent at compound interest? *Ans.* $3054.47.

6. What would $463.375 amount to in 13½ years, at 6 per cent, if the interest were compounded *semi-annually?* *Ans.* 1029.27.

7. In what time would $100, or any other principal, double itself, at 6 per cent compound interest? *Ans.* 11.9 years.

8. At what rate per cent would $100, or any other principal, double itself in 20 years, at compound interest? *Ans.* 3.526 per cent.

9. On the birth of a son, the father wishes to know what sum, at interest, at 7 per cent, if the interest be compounded semi-annually, would amount to $5000, by the time his son shall be 21 years of age. *Ans.* $1178.93.

Problems on Population.

(332.) Problems relating to the *natural* increase of population in a country, are analogous to those which occur in Compound Interest; and may be solved by the same formulas.

By the *natural increase*, is meant the excess of the number of *births* above the number of *deaths* among the inhabitants.

Other sources of increase in population are *immigration* from other countries, and the natural increase on the immigration.

10. The white population of the United States would double, from the natural increase alone, about every 27 years. What is the *annual percentum* of this increase?

We may assume any amount of population, and solve the problem by formula (3). *Ans.* 2.6004 per cent.

11. In point of fact, the white population of the United States doubles about every 23 years. What is the annual percentum of increase? *Ans.* 3.059 per cent.

12. The population of France, in the year 1821, was about 30 462 000, and in the year 1851, about 35 782 000; what was the annual percentum of increase during the 30 years? and in what time would the population be doubled at the same rate?

Ans. .537 per cent; 129.4 years.

13. The population of England in the year 1821 was about 11 261 000, and in the year 1851, about 17 906 000; what was the annual percentum of increase during the 30 years? and in what time would the population be doubled at the same rate?

Ans. 1.557 per cent; 44.83 years.

14. The whole population of the United States in the year 1790, when the first census was taken, was about 3 930 000, and in 1850, it was about 23 257 000. What has been the *decennial percentum* of increase? *Ans.* 34.491 per cent.

ANNUITIES.

(333.) An ANNUITY is properly a sum of money which is payable *annually;* but the term annuity is also applied to a sum which is to be paid semi-annually, quarterly, or at any regular intervals.

Pensions, salaries, rents, &c., are of the nature of annuities.

A *Perpetuity* is a *perpetual* annuity; that is, an annuity which is unlimited in duration, and is thus said to continue forever.

Of this kind may be the interest which a government pays on borrowed money.

An Annuity *in possession* is one of which the periods of payment have already commenced.

An Annuity *in reversion* is one of which the periods of payment are to commence at a specified future time, or on the occurrence of a specified future event.

An Annuity *in arrears* is one on which the periodical payments not having been made; these payments and the interest on them make up an Amount due on the annuity.

The first problem, which this subject presents, is that of finding the

Amount of an Annuity

(334.) Allowing Compound Interest—as is usually done in computations relating to annuities—

Let P represent the annual *payment;*

r the *interest* of $1 for 1 year;

n the number of *years* for which the annuity is computed.

A the whole *Amount* of the annuity:—then,

Considering that the several payments are due at the *terminations* of the successive years; the last payment P, must be taken without interest; the *second* payment from the last must be taken with 1 year's interest; the *third* payment from the last must be taken with 2 years' *compound* interest; and so on to the first payment, which must be taken with $n-1$ year's compound interest.

The *sum of all these parts* is the whole amount of the annuity; hence we have—

$$A=P+P(1+r)+P(1+r)^2+\cdots+P(1+r)^{n-1}.$$

The right hand member of this equation, is a *geometrical progression* whose ratio is $(1+r)$; and by taking the *sum of all its terms* (190), we have—

$$A=\frac{P(1+r)^n-P}{r}=\frac{P\times\overline{(1+r)^n-1}}{r}.$$

(335.) *Cor.* If the sum P were annually put at compound interest, for n successive years, the whole amount accumulated in the n years would be

$$A=P(1+r)+P(1+r)^2+\cdots+P(1+r)^n,$$

$$=\frac{P(1+r)^{n+1}-P(1+r)}{r},\ (190),$$

$$=\frac{P(1+r)\times\overline{(1+r)^n-1}}{r}.$$

Present Value of an Annuity.

(336.) The *Present Value* of an annuity for n years, is that sum which, at compound interest during the n years, would produce the Amount of the Annuity.

Let V represent the Present Value of the annuity, and P the annual *payment* on the annuity; then

By putting the Amount of V, at compound interest for n years (331), equal to the Amount of the annuity for the same time (334), we have

$$V(1+r)^n=\frac{P\times\overline{(1+r)^n-1}}{r};$$

and hence $$V=\frac{P\times\overline{(1+r)^n-1}}{r(1+r)^n}.$$

(337.) The Present Value of a *perpetuity* is that sum whose annual *interest* is equal to the annual *payment* on the perpetuity; and will therefore be found by dividing said payment by the interest of $1 for *one year*.

Hence, $\frac{P}{r}$ expresses the Present Value of a *perpetuity*.

Present Value of an Annuity in reversion.

(338.) The Present Value of an annuity *in reversion*, is that sum which, at compound interest until the commencement of the annuity, would amount to the *value of the annuity at that time.*

Suppose the annuity P is to commence at the expiration of t years, and continue for n years, and

Let S represent the Present Value of the annuity; then, putting the amount of S for t years (331), equal to the value of the annuity at the commencement of the n years (336), we have

$$S(1+r)^t = \frac{P \times \overline{(1+r)^n - 1}}{r(1+r)^n};$$

$$\text{and hence } S = \frac{P \times \overline{(1+r)^n - 1}}{r(1+r)^{n+t}}.$$

(339.) The Present Value S of a perpetuity P, *in reversion* for t years, may be found from the Equation

$$S(1+r)^t = \frac{P}{r}, (337);$$

$$\text{which gives } S = \frac{P}{r(1+r)^t}.$$

☞ The preceding Formulas become applicable to Annuities which are payable at *any regular intervals*, by taking r equal to the interest of $1 for one of those intervals.

EXAMPLE.

To find the Present Value of an annuity of $100 per annum for 3 years, commencing at the expiration of 4 years from the present time, allowing compound interest at 6 per cent.

By substituting the proper numbers for the letters P, r, n, t, in the formula already established, (338), the expression for the Present Value of the annuity is

$$\frac{100 \times \overline{(1.06)^3-1}}{.06(1.06)^7}.$$

The computation of the numerator of this expression would not be facilitated by the application of *logarithms*.

$$\underline{100 \times \overline{(1.06)^3-1}}=19.1016.$$

The logarithm of the denominator is

$$\text{log. of } .06+7\times\text{log. } 1.06=\bar{2}.778151+0.177142 \text{ (308), (310);}$$
$$=\bar{2}.955293;$$

and the corresponding natural number is .09021.

Hence, the required Present Value is

$$19.1016\div.09021=\$211.74.$$

Problems in Annuities.

1. An annuity of $300 per annum has remained unpaid for 25 years. What amount is due on the annuity, allowing compound interest at 6 per cent? *Ans.* $16459.50.

2. What amount is due on a pension of $500 a year, which has remained unpaid for 13 years, allowing the interest to be compounded at the rate of 5 per cent? *Ans.* $8856.30.

3. What would be the difference between the amount of $100 annually, put at compound interest, for 10 years, and the amount of an annuity of $100 a year, for the same time, supposing the rate of interest to be 6 per cent? *Ans.* $79.08.

4. Suppose $300 to be put at compound interest, and that to the stock is annually added 20; What will be the whole amount at the expiration of 6 years, the interest being 4 per cent? *Ans.* $492.24.

5. The annual rent of an estate is \$500. What would be the present value of the rents for 7 years, if 6 per cent compound interest be allowed? *Ans.* \$2791.19.

6. What is the present value of a perpetuity of \$5000 per annum, allowing the rate of interest to be 7 per cent? *Ans.* \$71428.57.

7. An annuity of \$1000 a year is to continue for 15 years, but is not to commence till the expiration of 10 years from the present time. What is the present value of the annuity, if compound interest be allowed at 6 per cent? *Ans.* \$5423.22.

8. A perpetuity of \$300 per annum is in reversion for 20 years. What is the present value, reckoning compound interest at 5 per cent? *Ans.* \$2261.36.

9. Which would be the most advantageous, a term of 15 years in an estate of \$1200 per annum, or the reversion of the same estate forever, after the expiration of that time; compound interest being reckoned at 6 per cent? *Ans.* The first, by \$3309.40.

10 A gentleman bequeathed to his widow, for five years, an estate worth \$500 per annum; to his daughter, the reversion of the same estate for the next 10 years; and to his son, the reversion of the estate forever afterwards. What was the value, in ready money, of each bequest, allowing compound interest at 6 per cent?
Ans \$2106.16; \$2749.95; \$3477.20

CHAPTER XV.

INFINITE AND INFINITESIMAL QUANTITIES.

VANISHING FRACTIONS.

(340.) An INFINITE quantity is one which is supposed to be *greater* than any assignable quantity; and an INFINITESIMAL quantity is one which is supposed to be *less* than any assignable quantity.

As thus defined, one *infinite* quantity may be *greater* than another; and one *infinitesimal* quantity may be *less* than another.

If to each of the digits, 1, 2, 3, &c., 0's were annexed without end, each of the resulting numbers

$$100000\ \&c.,\quad 200000\ \&c.,\quad 300000\ \&c.,$$

would be *infinite;* but the second number would be *double* the first, the third would be *three times* the first, and so on.

Again, if we take the *reciprocals* of the preceding infinites, we shall have the *infinitesimals*

$$\frac{1}{100000\ \&c.},\quad \frac{1}{200000\ \&c.},\quad \frac{1}{300000\ \&c.};$$

but the second of these infinitesimals would be *one-half* of the first, the third would be *one-third* of the first, and so on (147).

An infinitesimal may always be regarded as the *reciprocal* of an *infinite* quantity referred to the same *measuring unit.*

The character ∞ has been adopted to represent *infinity;*
hence $\frac{1}{\infty}$ will represent an *infinitesimal.*

The preceding illustrations show that two *infinite* quantities, or two *infinitesimals,* may have an *assignable ratio,* the one to the other; although of those quantities themselves no definite conceptions can be formed—the one kind being *illimitably great,* and the other illimitably *small.*

Of 0 *as representing an Infinitesimal.*

(341.) An infinitesimal differs from absolute *nothing* by less than any assignable quantity. 0 may therefore be taken as practically equivalent to the symbol $\frac{1}{\infty}$.

But one infinitesimal may be *less* than another; and therefore 0 denoting one infinitesimal may be less than 0 denoting another infinitesimal. This consideration will obviate some inconsistencies which would otherwise appear in the results of some of the calculations in which infinitesimals are employed.

Like inconsistencies will be precluded in calculations involving *infinite* quantities, by considering that these quantities are not all equal to one another.

CALCULUS OF INFINITE AND INFINITESIMAL QUANTITIES.

(342.) If any *finite* quantity be multiplied by *infinity*, the product will be *infinite*.

$$a \times \infty = \infty.$$

For, if the product were not infinite, the quotient found by dividing it by the *finite* factor a would not be infinite, as it should be, since it should be the other factor ∞.

(343.) If any *finite* quantity be multiplied by an *infinitesimal*, the product will be an *infinitesimal*.

$$a \times 0 = 0.$$

For, if the product were any *finite* quantity, or infinite, the quotient found by dividing it by the finite factor a would not be an infinitesimal, as it should be, to be equal to the other factor 0.

(344.) If any *finite* quantity be divided by *infinity*, the quotient will be an *infinitesimal*.

$$\frac{a}{\infty} = 0.$$

It is evident that the quotient will not be *infinite;* and it cannot be any *finite* quantity, for then the quotient multiplied by the divisor ∞, would not produce the *finite* dividend a (342).

Cor. Any *finite* quantity connected by $+$ or $-$ to *infinity*, may be canceled, since its *ratio* to ∞ is .0.

Thus, $\infty \pm a = \infty$; just as $a \pm 0 = a$.

(345.) If *infinity* be divided by any *finite* quantity, the quotient will be *infinite*.

$$\frac{\infty}{a} = \infty.$$

This follows from the equation, $a \times \infty = \infty$ (342), by dividing both sides by a.

(346.) If an *infinitesimal* be multiplied by *infinity*, the product will be a *finite* quantity.

$$0 \times \infty = a.$$

This follows from the Equation, $\frac{a}{\infty} = 0$ (344), by multiplying both sides by ∞.

(347.) If an *infinitesimal* be divided by any *finite* quantity, the quotient will be an *infinitesimal*.

$$\frac{0}{a} = 0.$$

This follows from the Equation $a \times 0 = 0$ (343), by dividing both sides by a.

(348.) If any *finite* quantity be divided by an *infinitesimal*, the quotient will be *infinite*.

$$\frac{a}{0} = \infty.$$

This follows from the Equation, $0 \times \infty = a$ (346), by dividing both sides by 0.

(349.) If *infinity* be divided by *infinity*, or an *infinitesimal* by an *infinitesimal*, the quotient will be a *finite* quantity.

$$\frac{\infty}{\infty} = a; \text{ or, } \frac{0}{0} = a.$$

These results follow from the Equation, $a \times \infty = \infty$ (342), and the Equation, $a \times 0 = 0$ (343), by dividing the former equation by ∞, and the latter by 0.

(350.) An infinitesimal with any *finite exponent*, is equal to an *infinitesimal;* but with an *infinitesimal* exponent, it is equal to *unity*

$$0^a = 0\text{; but } 0^0 = 1.$$

The product of any number of infinitesimals, or 0's, multiplied together, can be only 0 ; hence any *power* or *root* of 0 is 0.

To prove that $0^0 = 1$, it is only necessary to remark, that as any quantity whatever with exponent 0 is equal to *unity* (48), we may suppose that quantity to be an *infinitesimal.*

A more significant representation of the property $0^0 = 1$, is

$$\left(\frac{1}{\infty}\right)^{\frac{1}{\infty}} = 1.$$

In further illustration of the same property, let us take a series of decreasing fractions, with that root of each which is denoted by its denominator (329).

We shall find

$$\left(\frac{1}{10}\right)^{\frac{1}{10}} = \frac{1}{1.2589};$$

$$\left(\frac{1}{100}\right)^{\frac{1}{100}} = \frac{1}{1.0471};$$

$$\left(\frac{1}{1000}\right)^{\frac{1}{1000}} = \frac{1}{1.0069};$$

$$\left(\frac{1}{10000}\right)^{\frac{1}{10000}} = \frac{1}{1.0009}.$$

Here we find that as the denominators of the assumed fractions increase, the denominators of their *roots* approach nearer and nearer to *unity*. We conclude, therefore, that if the assumed denominator were *infinite*, the denominator of the corresponding *root* would be *unity;* hence

$$\left(\frac{1}{\infty}\right)^{\frac{1}{\infty}} = \frac{1}{1} = 1\text{; or } 0^0 = 1.$$

Connected with the preceding theory is the subject of

VANISHING FRACTIONS.

(351.) A *Vanishing Fraction* is one which takes the form $\frac{0}{0}$ for a particular value of a variable quantity in its numerator and denominator.

In the state of $\frac{0}{0}$ the fraction would appear to have *no determinate* value, since the quotient of $0 \div 0$ may be *any finite quantity* (349); but the subject to which the fraction refers may impose upon the form $\frac{0}{0}$ a *determinate value*, or else make it equal to 0 or ∞.

For example. The *sum* of 5 terms of a geometrical progression whose *first* term is a, and ratio r, is expressed by the fraction

$$\frac{ar^5-a}{r-1} \ (190).$$

If we now suppose the ratio r to be *unity*, this fraction will become

$$\frac{a-a}{1-1} = \frac{0}{0}, \text{ the symbol of } \textit{indeterminateness}.$$

But the sum of 5 terms of the progression, whose first term is a, and ratio *unity*, is $a+a+a+a+a=5a$, which is therefore the value of the fraction when its numerator and denominator are in a *vanishing state*.

The sum $5a$ may be found by dividing the assumed numerator by its denominator, and then making r in the quotient equal to *unity*.

$$r-1\ )\,ar^5-a\,(\ ar^4+ar^3+ar^2+ar+a.$$

When $r=1$, the quotient becomes $a+a+a+a+a=5a$.

By this method, the value of the fraction when $r=1$, is computed, without the vanishing of its terms; and to prevent the disappearance of *both the terms*, is an object to be attained in all computations for the values of these fractions. The preceding example is included among

Fractions which vanish in consequence of a Vanishing Factor in both their terms.

(352.) The value of a fraction which takes the form $\frac{0}{0}$ in consequence of its two terms containing *a common factor* that has become 0, for a particular value of the variable, may be found by first *canceling that factor*, and then resuming the supposed value of the variable.

If, in the Fractions

$$\frac{2(x-4)^2}{3x(x-4)^2}, \quad \frac{2(x-4)^3}{3(x-4)^2}, \quad \frac{2x(x-4)^2}{3(x-4)^3},$$

we suppose that $x=4$, the factor $(x-4)^2$, which is common to the several numerators and denominators, will be equal to 0 (350); and, with this factor retained, each of the three fractions would become $\frac{0}{0}$ (343).

By dividing each numerator and denominator by $(x-4)^2$— and then supposing $x=4$, we shall find

$$\frac{2(x-4)^2}{3x(x-4)^2} = \frac{2}{3\times 4} = \frac{2}{12} = \frac{1}{6},$$

$$\frac{2(x-4)^3}{3(x-4)^2} = \frac{2(x-4)}{3} = \frac{2\times 0}{3} = 0,$$

$$\frac{2x(x-4)^2}{3(x-4)^3} = \frac{2\times 4}{3(x-4)} = \frac{8}{3\times 0} = \infty.$$

From this example, we see that the Vanishing Fraction will be equal to a *determinate* quantity, 0, or ∞, according as the exponent of the common vanishing factor in the numerator, is *equal to*, greater, or *less*, than its exponent in the denominator.

It will serve to elucidate the nature of a Vanishing Fraction, if we consider its numerator and denominator as *continually approaching* 0, when the variable which they contain continually approaches a certain definite value; and regard the value of the fraction in its *vanishing state* as the ratio between *two infinitesimals.*

If we take the Fraction

$$\frac{1-x^2}{1-x},$$

which vanishes when $x=1$, and attribute to x the values $\frac{1}{2}$, $\frac{3}{4}$, $\frac{9}{10}$, $\frac{99}{100}$, successively, the results will be as follows:—

When $x=\frac{1}{2}$, $\frac{1-x^2}{1-x} = 1\frac{1}{2}$;

" $x=\frac{3}{4}$, $\frac{1-x^2}{1-x} = 1\frac{3}{4}$;

$$\text{when } x=\tfrac{9}{10}, \qquad \frac{1-x^2}{1-x} = 1\tfrac{9}{10},$$

$$\text{" } x=\tfrac{99}{100}, \qquad \frac{1-x^2}{1-x} = 1\tfrac{99}{100}.$$

We find that as the value of x approaches nearer to 1, the value of the fraction approaches nearer to 2; and from the *law of approximation* which is plainly developed, we may conclude that when the value of x is *infinitely near* 1, the value of the fraction is *infinitely near* 2.

The numerator and denominator will then be *infinitesimals*, but the value of the former will be *double that of the latter.*

To compute the value of the assumed fraction in its vanishing state, we have only to divide the numerator by the denominator, and then make $x=1$.

$$(1-x)^2 \div (1-x) = 1+x, \text{ which} = 2, \text{ when } x=1.$$

Different Values of the symbol $\frac{0}{0}$.

(353.) I. When $\frac{0}{0}$ is taken in a general sense, and not as resulting from any particular calculation, its value is *indeterminate;* since any *finite* quotient whatever × the divisor 0, will produce the dividend 0.

II. When $\frac{0}{0}$ expresses the value of the *unknown* quantity in an *Identical Equation*, it is *indeterminate*, since any quantity whatever will satisfy such an equation. (See Example II, page 206.)

III. When $\frac{0}{0}$ has been derived from a fraction whose *terms vanish* for a particular value of a variable quantity contained in them; its value, as before shown, will be a *determinate* quantity, or else it will be 0 or ∞ (352).

☞ The purposes of common Algebra do not require any further notice of Vanishing Fractions; the subject is chiefly important in its connection with the higher mathematics, and its general exposition will be found in treatises on the *Differential Calculus.*

EXERCISES

On Vanishing Fractions.

1. Find the value of the fraction

$$\frac{3x-9}{x^2-7x+12}, \text{ when } x=3.$$

In its present form, the fraction reduces to $\frac{0}{0}$, when x is made equal to 3.

By decomposing both the numerator and denominator—the general method of doing which is to divide them by their greatest *common measure*—the given fraction becomes

$$\frac{3(x-3)}{(x-4)(x-3)}.$$

Canceling the common factor $x-3$, and then making $x=3$, we find the value of the fraction in its vanishing state to be

$$\frac{3}{3-4}=\frac{3}{-1}=-3.$$

2. Find the value of the fraction

$$\frac{5(x-2)^2}{3x^3-12x^2+12x}, \text{ when } x=2. \qquad \textit{Ans. } \frac{5}{6}.$$

3. Find the value of the fraction

$$\frac{x^3-ax^2-a^2x+a^3}{x^2-a^2}, \text{ when } x=a. \qquad \textit{Ans. } 0.$$

4. Find the value of the fraction

$$\frac{ax^2+ac^2-2acx}{bx^2-2bcx+bc^2}, \text{ when } x=c. \qquad \textit{Ans. } \frac{a}{b}.$$

5. Find the value of the fraction

$$\frac{ax-x^2}{a^4-2a^3x+2ax^3-x^4}, \text{ when } x=a. \qquad \textit{Ans. } \infty.$$

CHAPTER XVI.

INFINITE SERIES.—DIFFERENTIAL METHOD OF SERIES.—INTERPOLATION OF SERIES.

INFINITE SERIES.

(354.) An Infinite Series is a succession of an infinite number of terms, in accordance with some determinate *law*, different in different series, and discoverable from a few of the leading terms of the series.

The *General Term* of a Series is an expression depending for its form on the law of the series, and supposed to vary in the values of one or more of its parts, so as to represent, successively, every term of the series.

A *converging* Infinite Series is one whose successive terms continually diminish, and in which the *sum of all the terms* is a *finite quantity*.

If we reduce the fraction $\frac{2}{3}$ to a decimal, we find

$$\frac{2}{3} = .66666, \&c.$$

$$= \frac{6}{10} + \frac{6}{100} + \frac{6}{1000} + \frac{6}{10000} + \frac{6}{100000} + \&c.;$$

which is a *converging* infinite series; the *law* of the series being, that each succeeding term is $\frac{1}{10}$ of the next preceding term; and the *sum* of the whole series continued *ad infinitum*, is evidently the fraction $\frac{2}{3}$ from which the series was derived.

The *general term* of this series is $\frac{6}{10^n}$; because if we make the exponent n successively $=1, 2, 3$, &c., every term of the series will be thus represented.

A *diverging* Infinite Series is one whose successive terms continually increase; the sum of all the terms being *infinite;* as the series of numbers

1, 2, 3, 4, 5, &c, *ad infinitum*.

EXPANSION OF SERIES.

(355.) For the purposes of mathematical investigation it is sometimes expedient that an expression be developed, or *expanded*, into a Series. Of this, an example has occurred in the Computation of Logarithms by means of a *converging* series (314); but such instances are most frequent in the higher branches of Mathematics.

The Algebraic methods of Expanding Series are by Division, Evolution, the Binomial Theorem, and the method of Assumed Coefficients—only the last two of which need a particular explanation.

Series by the Binomial Theorem.

(356.) This Theorem, as formerly said (204), explains a general method of developing a Binomial according to any exponent with which the binomial may be affected.

When the exponent is a *positive integer*, the development will terminate at that term in which the exponent of the leading quantity is, by the continual subtraction of a *unit*, reduced to 0 (205).

But when the exponent of the Binomial is a proper *fraction*, or is *negative*, the exponent of the leading quantity in the development can never become 0, and the development will, accordingly, run into an *Infinite Series.*

EXAMPLE I.

To expand $(a-x)^{\frac{1}{2}}$ into an Infinite Series.

According to the Binomial Theorem (204 ... 1 ... 2), the *literal* factors, without their coefficients, are

$$a^{\frac{1}{2}},\ a^{-\frac{1}{2}}x,\ a^{-\frac{3}{2}}x^2,\ a^{-\frac{5}{2}}x^3,\ a^{-\frac{7}{2}}x^4,\ \&c.$$

The *coefficients*—with their component parts arranged so as to show the *law* which determines them (204...3)—are

$$1,\quad \frac{1}{2},\quad \frac{1}{2}\times-\frac{1}{2}\div2=-\frac{1}{2.4},\quad -\frac{1}{2.4},\times-\frac{3}{2}\div3=\frac{1.3}{2.4.6},$$

$$\frac{1.3}{2.4.6}\times-\frac{5}{2}\div4=-\frac{1.3.5}{2.4.6.8},\ \&c.$$

In the general development of $(a-x)^n$, the *signs* of the terms are alternately + and − (204 . . . 4). By applying these signs to the coefficients just found, and annexing the literal factors, we have

$$a^{\frac{1}{2}} - \frac{1}{2}a^{-\frac{1}{2}}x - \frac{1}{2.4}a^{-\frac{3}{2}}x^2 - \frac{1.3}{2.4.6}a^{-\frac{5}{2}}x^3 - \frac{1.3.5}{2.4.6.8}a^{-\frac{7}{2}}x^4, \&c.;$$

$$\text{or } a^{\frac{1}{2}} - \frac{x}{2a^{\frac{1}{2}}} - \frac{x^2}{8a^{\frac{3}{2}}} - \frac{x^3}{16a^{\frac{5}{2}}} - \frac{5x^4}{128a^{\frac{7}{2}}}, \&c. \ (78).$$

We have here the Square Root of the binomial $a-x$, in an Infinite Series. The same Series might also be obtained by the common Rule for extracting the square root of a Polynomial.

EXAMPLE II.

To expand $\frac{2}{a+x}$, or $2(a+x)^{-1}$ (78), into an Infinite Series.

Applying the Binomial Theorem to the factor $(a+x)^{-1}$, the *literal* factors, without their coefficients, are

$$a^{-1},\ a^{-2}x,\ a^{-3}x^2,\ a^{-4}x^3,\ \&c.$$

The *coefficients*—with the manner of finding them—are

$$1,\ -1,\ -1\times-2\div2=1,\ 1\times-3\div3=-1,\ -1\times-4\div4=1,\ \&c.$$

Hence we find

$$\frac{2}{a+x} = 2(a^{-1}-a^{-2}x+a^{-3}x^2-a^{-4}x^3+\ \&c).$$

$$= \frac{2}{a} - \frac{2x}{a^2} + \frac{2x^2}{a^3} - \frac{2x^3}{a^4} + \ \&c.$$

The same Series may be obtained by *Division.*

The expansion of a Binomial will sometimes be facilitated by resolving the quantity into a monomial and a binomial factor, and applying the Theorem to the latter. This is shown in

EXAMPLE III.

To expand $(2x-x^2)^{\frac{1}{3}}$ into an Infinite Series.

The given binomial may be resolved into

$$\sqrt[3]{2x}\left(1-\frac{x}{2}\right)^{\frac{1}{3}}\ (234).$$

By applying the Binomial Theorem to the factor

$$\left(1-\frac{x}{2}\right)^{\frac{1}{3}},$$

reducing the series to its simplest form, and prefixing the factor $\sqrt[3]{2x}$, we shall find

$$(2x-x^2)^{\frac{1}{3}}=\sqrt[3]{2x}\left(1-\frac{x}{6}-\frac{x^2}{36}-\frac{5x^3}{648}-\text{\&c.}\right)$$

The Series in this example may also be found by extracting the cube root of the factor $1-\frac{x}{2}$, according to the Rule for the Cube Root of a Polynomial.

Series by Assumed Coefficients.

(357). An algebraic expression, which is a function of x (284), may be put equal to a Series in the ascending powers of x, commencing with x^0, with *assumed coefficients* which are independent of x.

These coefficients of the powers of x may then be determined—by clearing the equation of the *fraction*, or the *surd*, which may be found in the first member—and *equating the coefficients* of the *like powers* of x in the two members of the equation.

This method of expanding Series depends on the following Theorem:

(358.) In every Equation of the form

$$A+Bx+Cx^2+Dx^3+\text{ \&c.}=K+Lx+Mx^2+Nx^3+\text{ \&c.},$$

supposed to be true for *all values* of x; the coefficients of the *like powers* of x are respectively equal to each other.

Since the equation is supposed to be true for all values of x, it must be true when $x=0$. This value of x causes all the terms in the equation to vanish, except A and K; and therefore

$$A=K.$$

By suppressing these two equal terms, the assumed equation becomes

$$Bx+Cx^2+Dx^3+\text{ \&c.} = Lx+Mx^2+Nx^3+\text{ \&c.}$$

Dividing each side of this Equation by x, we have

$$B+Cx+Dx^2+\text{ \&c.} = L+Mx+Nx^2+\text{ \&c.}$$

Again supposing $x=0$, all the terms vanish except B and L; and hence

$$B=L.$$

In the same manner, it may be shown that

$$C=M;\ D=N;\text{ \&c.}$$

☞ In the assumed Equation, the terms A and K, may be understood to be Ax^0 and Kx^0; the factor x^0 being omitted, without affecting the values of the terms containing it (48); and

Whenever, in the application of this method, any of the inferior powers of x are wanting in the first member of the Equation—such powers must be supplied with the coefficient 0. The coefficients of the same powers in the second member will then be equal to 0.

EXAMPLE I.

To expand $\dfrac{1-x^2}{1+x^2-x^4}$ into an Infinite Series.

Assuming the given fraction to be equal to a Series in the ascending powers of x with coefficients that are independent of x; we have

$$\frac{1-x^2}{1+x^2-x^4} = A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+Gx^6+\text{ \&c.}$$

By clearing the Equation of the fraction in the first member, and placing the like powers of x, in the second member, one under another we shall find

$$\begin{array}{rl} 1-x^2= & A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+Gx^6+\text{ \&c.} \\ & \qquad\quad\ \ +Ax^2+Bx^3+Cx^4+Dx^5+Ex^6+\text{ \&c.} \\ & \qquad\qquad\qquad\qquad\ -Ax^4-Bx^5-Cx^6-\text{ \&c} \end{array}$$

By supplying $0x$ in the first member, and equating the coefficients of the like powers of x in the two members; we have

$$\begin{array}{rlrl}
A=1; & & & \\
B=0; & & & \\
C+A=-1, & \text{which gives} & C=-1-A= & -2; \\
D+B=0, & \text{"} \quad \text{"} & D=0-B= & 0; \\
E+C-A=0, & \text{"} \quad \text{"} & E=A-C= & 3; \\
F+D-B=0, & \text{"} \quad \text{"} & F=B-D= & 0; \\
G+E-C=0, & \text{"} \quad \text{"} & G=C-E= & -5; \ \&c.
\end{array}$$

The values of the coefficients A, B, C, &c., have thus been determined, and by substituting these values in the first equation, we have

$$\frac{1-x^2}{1+x^2-x^4}=1-2x^2+3x^4-5x^6+\ \&c.$$

The *law* of this Series is, that its terms are alternately + and — ; that the exponents of x increase by 2, continually, beginning with x^0; and that the sum of the coefficients of any two consecutive terms, is equal to the coefficient of the following term.

The same Series may be readily obtained by Division.

There are certain forms of algebraic expressions which cannot be expanded by the method of Assumed Coefficients; and the impossibility of such expansion will be shown by some impossibility or absurdity in the results obtained by this method.

This is seen in

EXAMPLE II.

To expand $\frac{1}{3x-x^2}$ into an Infinite Series.

By proceeding as in the former Example, we should find

$$1=3Ax+(3B-A)x^2+(3C-B)x^3+\ \&c.$$

The 1 in the first member is $1x^0$; and x^0 must be supplied in the second member, with the coefficient 0.

Hence, by equating the coefficients of the like powers of x, in the two members, we should have

$$1=0;\ 3A=0;\ 3B-A=0;\ \&c.$$

Now, the result $1=0$, is *impossible;* and hence the given fraction cannot, under its present form, be expanded by this method.

But the given fraction may be resolved into $\frac{1}{x} \times \frac{1}{3-x}$; and the *second of these two factors* may be expanded into

$$\frac{1}{3}+\frac{x}{9}+\frac{x^2}{27}+\frac{x^3}{81}+\text{ \&c.}$$

Multiplying this series by the first of the two factors, we find

$$\frac{1}{3x-x^2}=\frac{1}{3x},\text{ or }\frac{x^{-1}}{3},+\frac{x^0}{9}+\frac{x^1}{27}+\frac{x^2}{81}+\text{ \&c.}$$

In this example, as well as in the preceding one, the expansion of the Series might have been effected by Division. In the following Example, the present method is applied to the expansion of the Square Root.

EXAMPLE III.

To expand $(a^2-2x^2)^{\frac{1}{2}}$ into an Infinite Series.

By squaring each side of the Equation

$$(a^2-2x^2)^{\frac{1}{2}}=A+Bx+Cx^2+Dx^3+Ex^4+\text{\&c.},$$

with a view to rationalizing the first member—we find

$$\begin{aligned}a^2-2x^2=A^2+2ABx&+2ACx^2+2ADx^3+2AEx^4\\ &+\;B^2x^2\;+2BCx^3+2BDx^4\\ &\qquad\qquad\qquad\quad+\;C^2x^4+\text{\&c.}\end{aligned}$$

By equating the coefficients of the like powers of x, we have

$A^2=a^2$; $2AB=0$; $2AC+B^2=-2$; $2AD+2BC=0$; &c.; from which we shall find

$$A=a;\ B=0;\ C=-\frac{1}{a};\ D=0;\ E=-\frac{1}{2a^3},\text{ \&c.};$$

and by substituting these values in the first equation, we find

$$(a-2x)^{\frac{1}{2}}=a-\frac{x^2}{a}-\frac{x^4}{2a^3}-\text{ \&c.}$$

This Series might be found by the Binomial Theorem, or the Rule for extracting the square root of a polynomial.

A useful application of the method of Assumed Coefficients, is that by which certain fractions may be separated into

Component Partial Fractions.

(359.) A fraction with a *composite denominator*, may be separated into component *partial* fractions—by assuming a numerator for each of the factors of the given denominator—and then determining these numerators by equating the *sum of the products* of each of them × all the partial denominators except its own, with the *numerator of the given fraction.*

EXAMPLE.

To divide $\frac{2x+37}{x^2+x-20}$ into component *partial fractions.*

The factors of the given denominator—the most general method of finding which, would be to solve the equation formed by putting the denominator equal to 0 (253),—are

$$x+5, \text{ and } x-4.$$

Assuming A and B as *numerators* to these factors, we have

$$\frac{A}{x+5} \text{ and } \frac{B}{x-4}.$$

Reducing to a common denominator, and adding together these two fractions, we get

$$\frac{A(x-4)+B(x+5)}{x^2+x-20}.$$

For this to be equal to the given fraction, we must have

$$A(x-4)+B(x+5)=2x+37;$$

$$\text{or } (A+B)x+(5B-4A)=2x+37.$$

By applying the principle of equal coefficients of like powers of x (358), to the last equation, we have

$$A+B=2, \text{ and } 5B-4A=37;$$

from which we shall find $A=-3$, and $B=5$.

The given fraction may therefore be separated into the *partial fractions*

$$\frac{-3}{x+5} \text{ and } \frac{5}{x-4}.$$

By adding these two fractions together, it will be found that their sum is equal to the given fraction; and thus the operation will be verified.

EXERCISES

On the Expansion of Series, &c.

1. By the Binomial Theorem find the value of

$(1+x)^{\frac{1}{2}}$ in an infinite series.

Ans. $1+\frac{x}{2}-\frac{x^2}{2.4}+\frac{3x^3}{2.4.6}-\frac{3.5x^4}{2.4.6.8}$, &c.

2. By the Binomial Theorem find the value of

$(a^3-x)^{\frac{1}{3}}$, in an infinite series.

Ans. $a-\frac{x}{3a^2}-\frac{x^2}{9a^5}-\frac{5x^3}{81a^8}-\frac{10x^4}{243a^{11}}$, &c.

3. By the Binomial Theorem find the value of

$\frac{1}{(a^2+x)^{\frac{1}{2}}}$, or $(a^2+x)^{-\frac{1}{2}}$, in an infinite series

Ans. $\frac{1}{a}-\frac{x}{2a^3}+\frac{3x^2}{2.4a^5}-\frac{3.5x^3}{2.4.6a^7}+\frac{3.5.7x^4}{2.4.6.8a^9}$, &c.

4. By the method of Assumed Coefficients expand

$(1+x+x^2)^{\frac{1}{2}}$ into an infinite series.

Ans. $1+\frac{x}{2}+\frac{3x^2}{8}-\frac{3x^3}{16}+\frac{3x^4}{128}$, &c.

5. By the method of Assumed Coefficients expand

$\frac{1-x}{1-2x-3x^2}$ into an infinite series.

Ans. $1+x+5x^2+13x^3+41x^4+121x^5$, &c.

6. By the method of Assumed Coefficients separate

$\frac{2x+3}{x^3+x^2-2x}$ into component *partial* fractions.

Ans. $\frac{5}{3(x-1)}-\frac{1}{6(x+2)}-\frac{3}{2x}$.

SUMMATION OF INFINITE SERIES.

(360.) The SUMMATION of an Infinite Converging Series, consists in finding the sum of all the terms of the series.

A decreasing *Geometrical Progression*, continued without limit, is one species of Infinite Converging Series; and the method of finding its sum has already been given (191).

The sum of a *finite*, or of an *infinite* number of terms of various other classes of Converging Series, may be determined by the formula,

$$\frac{c}{v(v+d)}=\frac{1}{d}\left(\frac{c}{v}-\frac{c}{v+d}\right);$$

in which the fractions, excepting the coefficient of the second member, are the *general terms* of different series; and the two members will be found to be equal, by performing the operations which are indicated.

If now, by attributing a *constant* or a *variable* value to c, a variable value to v, and a constant value to d, in this formula; the *first member* shall produce the successive terms of a *given Series;* the *sum* of this series will be equal to the *difference* between the two new series

$$\frac{c}{v} \text{ and } \frac{c}{v+d}, \text{ multiplied by } \frac{1}{d}.$$

This is evident from considering, that each term of the given series, is, by the formula, equal to the *difference* between the two corresponding terms in the two new series, multiplied by $\frac{1}{d}$.

EXAMPLE.

To find the sum of n terms, and also of an infinite number of terms of the series

$$\frac{1}{1\times2}, \frac{1}{2\times3}, \frac{1}{3\times4}, \frac{1}{4\times5}, \text{\&c.}$$

If we make $c=1$, $d=1$, and $v=1, 2, 3, 4$, &c., in succession; the first member,

$$\frac{c}{v(v+d)},$$

of the preceding Formula will produce the successive terms of the given series; and the second member will produce, for the sum of n terms of the series,

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}$$
$$-\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}+\frac{1}{n+1}.\right)$$

This sum is equal to

$$1-\frac{1}{n+1}=\frac{n}{n+1}$$

And if n, the number of terms, be *infinite*, their sum will be

$$\frac{\infty}{\infty+1}=\frac{\infty}{\infty}=1\ (344\ldots\mathit{Cor.})$$

☞ The quotient of $\infty \div \infty$, without reference to the origin of this symbol in particular cases, is an *indeterminate* finite quantity (349); but when two *infinites*, as in the present example, are known to be equal to each other, their quotient is necessarily *unity*.

A given Series may sometimes, by a simple *multiplication*, or *division*, be reduced to the form of another series whose sum is determinable by the preceding Formula; and from this sum that of the given series may then be derived by a converse division or multiplication. Thus,

To find the sum of n terms of the series

$$\frac{1}{1},\ \frac{1}{3},\ \frac{1}{6},\ \frac{1}{10},\ \frac{1}{15},\ \&c.$$

Dividing this series by 2, we shall obtain the same series as that in the former example; in which the sum of n terms is

$$\frac{n}{n+1}.$$

The sum of n terms of the present series is therefore

$$\frac{n}{n+1}\times 2=\frac{2n}{n+1}.$$

If n, the number of terms, were infinite, their sum would be

$$\frac{2\times\infty}{\infty+1}=2.$$

EXERCISES

On the Summation of Infinite Series.

1. Find the sum of n terms, and also of an infinite number of terms, of the series

$$\frac{1}{1\times 3},\ \frac{1}{2\times 4},\ \frac{1}{3\times 5},\ \frac{1}{4\times 6},\ \&c.$$

Ans. $\frac{1}{2}\left(\frac{3}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right)$; and $\frac{3}{4}$.

2. Find the sum of n terms, and also of an infinite number of terms of the series

$$\frac{2}{1\times 3},\ \frac{2}{3\times 5},\ \frac{2}{5\times 7},\ \frac{2}{7\times 9},\ \&c.$$

Ans. $1-\frac{1}{2n+1}$; and 1.

3. Find the sum of n terms, and also of an infinite number of terms, of the series

$$\frac{1}{1\times 4},\ \frac{1}{2\times 5},\ \frac{1}{3\times 6},\ \frac{1}{4\times 7},\ \&c.$$

Ans. $\frac{1}{3}\left(\frac{11}{6}-\frac{1}{n+1}-\frac{1}{n+2}-\frac{1}{n+3}\right)$; and $\frac{11}{18}$.

4. Find the sum of n terms, and also of an infinite number of terms of the series

$$\frac{1}{1\times 3},\ -\frac{1}{2\times 4},\ +\frac{1}{3\times 5},\ -\frac{1}{4\times 6},\ \&c.$$

In applping the formula to this series, put c equal, successively, to 1, -1, $+1$, -1, &c.

Ans. The sum of n terms is $\frac{1}{2}\left(\frac{1}{2}\mp\frac{1}{n+1}\pm\frac{1}{n+2}\right)$;

in which the ambiguous sign ± is + when n is an *even* number, and — when n is an *odd* number; and the sum of an infinite number of terms of the given series is $\frac{1}{4}$.

DIFFERENTIAL METHOD OF SERIES.

(361.) The Differential Method of Series is a method of solving certain problems in series by means of successive *orders of differences* among the terms.

1. The *first order of differences* in a series is found by subtracting the first term of the series from the second, the second from the third, &c.

2. The *second order of differences* in a series is found by subtracting the first term of the *first order* of differences from the second term, the second from the third, &c.

3. The *third order of differences* in a series is found by subtracting the first term of the *second order* of differences from the second term, the second from the third, &c.

In like manner is found the *fourth order of differences*, &c.

Thus, in the Series of *squares*

1, 4, 9, 16, 25, 36, 49, 64, &c.,

the *first order of differences* is

3, 5, 7, 9, 11, 13, 15;

the *second order of differences* is

2, 2, 2, 2, 2, 2;

the *third order of differences* is

0, 0, 0, 0, 0.

The terms in the successive orders of differences are diminished by successive subtractions.

Thus, taking the right hand terms in the preceding example, the 15 is 64—49; the 2 is 15−13; the 0 is 2−2.

The successive orders of differences thus tend to 0; and will, in general, vanish at an order which depends on the law of the particular Series.

In the series of *cubes*

1, 8, 27, 64, 125, 216, &c.,

the *fourth order of differences* will be 0.

To find the n*th Term of a Series.*

(362.) If a denote the *first term* of a Series, and D', D'', &c., the first terms, respectively, of the successive *orders of differences*—the nth term of the Series will be equal to

$$a+(n-1)D'+\frac{(n-1)(n-2)}{2}D''+\frac{(n-1)(n-2)(n-3)}{2\times 3}D'''+\&c.,$$

the coefficients of the terms in this Formula being the same as in the $(n-1)$ power of a Binomial (204 ... 3).

For, let the Series be represented by

$$a,\ b,\ c\ d,\ e,\ \&c.$$

The *first order of differences* in this series is

$$b-a,\quad c-b,\quad d-c,\quad e-d,\quad \&c.;$$

the *second order of differences* is

$$c-2b+a,\quad d-2c+b,\quad e-2d+c,\quad \&c.;$$

the *third order of differences* is

$$d-3c+3b-a,\quad e-3d+3c-b,\quad \&c.;$$

the *fourth order of differences* is

$$e-4d+6c-4b+a,\ \&c.$$

Taking the *first* of these differences, we have the several Equations

$$D'=b-a$$
$$D''=c-2b+a$$
$$D'''=d-3c+3b-a$$
$$D''''=e-4d+6c-4b+a,\ \&c.$$

By transposition, substitution, &c., in these Equations, we shall find

$$b=a+D'$$
$$c=a+2D'+D''$$
$$d=a+3D'+3D''+D'''$$
$$e=a+4D'+6D''+4D'''+D'''',\ \&c.$$

Here the coefficients in the value of c, the 3*d* term of the original Series, are 1, 2, 1, the same as in the 2*d* power of a Binomial (204...3); the coefficients in the value of d, the 4*th* term of the series, are 1, 3, 3, 1, the same as in the 3*d* power of a Binomial; and so on.

Hence, the coefficients in the value of the nth term of the Series, are the same as in the $(n-1)$ power of a Binomial.

EXAMPLE.

To find the *twelfth* term of the Series

$$2,\ 6,\ 12,\ 20,\ 30,\ \&c.,$$

in which the succeeding terms are obtained by adding, successively, 4, 6, 8, 10, &c., to the preceding term.

The successive orders of differences in this series, are

$$4,\ 6,\ 8,\ 10,\ \&c.,$$
$$2,\ 2,\ 2,\ \&c.,$$
$$0,\ 0.$$

Hence, by making the proper substitutions for the letters a, n, D', &c., in the preceding Formula, the 12th term of the Series is found to be

$$2+11\times4+\frac{11\times10}{2}\times2=156.$$

To find the Sum of n *terms of a Series.*

(363.) If a denote the *first term* of a Series, and D', D'', &c., the first terms, respectively, of the successive *orders of differences,*—the sum of n terms of the Series will be equal to

$$na+\frac{n(n-1)}{2}D'+\frac{n(n-1)(n-2)}{2\times3}D''+\&c.$$

For, let the Series be represented by

$$a,\ b,\ c,\ d,\ e,\ \&c.,$$

and assume the new series

$$0,\ a,\ a+b,\ a+b+c,\ a+b+c+d,\ \&c.$$

The 3d term of the new series is the sum of the first *two terms* of the given series; the 4th term of the new series is the sum of the first *three terms* of the given series; and so on.

Hence, the $(n+1)$ term of the new series will be equal to the sum of n terms of the given series.

In the new series the first term of the *first order* of differences is a; the first term of the *second order* of differences will be found to be $b-a$; which is D' in the Formula of the preceding problem (362); the first term of the *third order* of differences will be found to be $c-2b+a$, which is D'' in that Formula; and so on.

If therefore, in said Formula, we substitute 0 for a, a for D', D' for D'', &c., and $n+1$ for n; we shall obtain a Formula, and it will be the one just stated, for the sum of n terms of a given series.

EXAMPLE.

To find the sum of 20 terms of the Series

1, 3, 6, 10, 15, &c.,

in which the succeeding terms are obtained by adding, successively, 2, 3, 4, 5, &c., to the preceding term.

The successive orders of differences are

2, 3, 4, 5, &c.,
1, 1, 1,
0, 0.

By making the proper substitutions for n, a, D', &c., in the Formula, the sum of 20 terms of the given Series is found to be

$$20\times 1+\frac{20\times 19}{2}\times 2+\frac{20\times 19\times 18}{2\times 3}=1540.$$

EXERCISES

On the Differential Method of Series.

1. Find the 20th term of the series

1, 3, 5, 7, &c. *Ans.* 39

2. Find the sum of 12 terms of the series

1, 4, 8, 13, 19, &c. *Ans.* 430.

3. Find the 20th term of the series

1, 3, 6, 10, 15, &c. *Ans.* 210.

4. Find the sum of 30 terms of the series

2, 6, 12, 20, 30, &c. *Ans.* 9920.

5. Find the nth term of the series

2, 12, 30, 56, 90, &c. *Ans.* $4n^2-2n$.

6. Find the sum of n terms of the series

1, 2, 3, 4, 5, 6, &c. *Ans.* $\frac{n^2+n}{2}$.

EXAMPLES IN THE PILING OF BALLS, &c.

1. *Triangular Pile.*

To find the number of Balls that may be piled on a *triangular base*—the number in *each* side of the base being n.

The pile will terminate in a single ball at the top; and the numbers in the successive *horizontal layers*, or strata, from the top downwards, will evidently be

$$1,\ 1+2,\ 1+2+3,\ 1+2+3+4,\ 1+2+3+4+5,\ \&c.;$$

and the number of *strata* will evidently be equal to the number n of balls in each *side* of the base.

The solution, therefore, consists in finding the sum of n terms of the series

$$1,\ 3,\ 6,\ 10,\ 15,\ 21,\ \&c.,\ (363).$$

The number of balls will therefore be

$$n+n(n-1)+\frac{n(n-1)(n-2)}{6}=\frac{n^3+3n^2+2n}{6}=\frac{n(n+1)(n+2)}{6}.$$

2. *Square Pile.*

To find the number of balls that may be piled on a *square base*—the number in *each side* of the base being n.

The pile will evidently terminate in a single ball at the top; and the numbers in the successive horizontal layers, or strata, from the top downwards, will evidently be

$$1,\ 2\times2,\ 3\times3,\ 4\times4,\ 5\times5,\ \&c.;$$

and the number of strata will evidently be equal to the number n of balls in each side of the base.

The solution consists then in finding the sum of n terms of the series

$$1,\ 4,\ 9,\ 16,\ 25,\ 36,\ \&c.,\ (363).$$

The number of balls will therefore be

$$n+\frac{n(n-1)}{2}\times3+\frac{n(n-1)(n-2)}{3}=\frac{n(n+1)(2n+1)}{6}.$$

3. *Rectangular Pile.*

To find the number of balls that may be piled on a *rectangular base;* the number in the shorter side of the base being n, and in the longer $n+m$.

Since a *square* pile terminates in a single ball at the top; it is plain that a *rectangular* pile terminates in a *row of balls*, the number of which is $1+$ the number by which the longer exceeds the shorter side of the base.

The top row therefore contains $m+1$ balls; the layer below it contains 2 rows of $m+2$ balls; the next layer 3 rows of $m+3$ balls; and so on; and the number of layers is equal to the number n of balls in the shorter side of the base.

The whole number of balls is therefore equal to

$$(m+1)+2(m+2)+3(m+3)+\ldots n(m+n);$$

which by development becomes

$$m+2m+3m+\ldots nm+1+2\times2+3\times3+\ldots n\times n$$
$$=m(1+2+3+\ldots n)+1+4+9+\ldots n^2.$$

The sum of the n terms in the parenthesis is

$$n+\frac{n(n-1)}{2}=\frac{n(n+1)}{2}\ (363);$$

and the sum of the n terms 1, 4, 9, &c., as found for the *square pile*, is

$$\frac{n(n+1)(2n+1)}{6};$$

so that the whole is equal to

$$\frac{mn(n+1)}{2}+\frac{n(n+1)(2n+1)}{6}=\frac{n(n+1)(3m+2n+1)}{6}.$$

☞ In solving Problems of this kind, it will be well to remember, that the number of horizontal *layers* in a *triangular*, or *square* pile, is always equal to the number of balls in each side of the base; and in a *rectangular* one to the number in the *shorter* side of the base.

6. How many balls are there in a triangular pile, each side of whose base contains 20 balls? and how many in a square pile, each side of whose base also contains 20? *Ans.* 1560, and 2870.

7. How many balls are there in a rectangular pile, the sides of whose base contain 20 and 30 balls respectively? *Ans.* 4970.

8. How many balls are there in an *incomplete* triangular pile of 10 layers, or strata,—each side of whose base contains 30 balls?
Ans. 3420.

9. How many balls are there in an incomplete square pile,—the number in each side of the base being 14, and in each side of the upper stratum 6? *Ans*. 960.

10. How many balls are there in an incomplete rectangular pile of 12 strata,—the longer side of the base containing 46, and the shorter 20 balls? *Ans*. 7190.

INTERPOLATION OF SERIES.

(364.) The INTERPOLATION OF A SERIES consists in finding an *intermediate term* in the series, which shall correspond to any proposed intermediate term in another series; the two given series having some determinate relation to each other.

For example, having given the series of *Logarithms* corresponding to the numbers 31, 32, 33, 34; to find, the logarithm of 33.75.

The terms of the Series in which the Interpolation is to be effected, are *functions* of the corresponding terms in the other series, because the values of the former depend on those of the latter; and the terms of the latter series are sometimes called the *arguments* of the corresponding functions.

Thus, the Logarithm of a number is *a function* of that number; while the number itself is the *argument* of the function.

Equidistant functions are such as have their corresponding arguments in *arithmetical progression.*

Thus, the Logarithms of the numbers 20, 25, 30, &c., are *equidistant* functions, since these numbers form an arithmetical progression.

In the usual applications of this subject, as in finding intermediate numbers between those which are given in Mathematical and Astronomical Tables, the series to be interpolated consists, for the most part, of *equidistant* functions.

When the differences of the *functions* are very nearly proportional to the differences of their *arguments*, the interpolation of a term may be effected by *first differences;* which is the method employed in finding the Logarithm of a number containing more than four figures (324).

When such proportion does not exist—as is the case with logarithms of small numbers—the interpolation may be effected by a method which regards also the higher orders of differences of the functions; and which may be called the

Differential Method of Interpolating a Series.

(365.) If x represent the term to be interpolated in a Series whose given terms are *equidistant; y* the number of *intervals* and *part* of an interval that the term x is removed from the first term a; and D', D'', &c., the first terms, respectively, of the successive *orders of differences;* then will

$$x=a+yD'+\frac{y(y-1)}{2}D''+\frac{y(y-1)(y-2)}{2\times 3}D'''+\&c.$$

For if n be the ordinal number of the term x, we shall have

$$y=n-1.$$

Then, by substituting y for $n-1$ in the Formula for the nth term of a series (362), we have the Formula of Interpolation, as above.

EXAMPLE.

In latitude 40°, one degree of longitude is 45.96 miles; in latitude 41°, 45.28 miles; in latitude 42°, 44.59 miles; in latitude 43°, 43.88 miles. How many miles make one degree of longitude where the latitude is 42° 30′?

The given series of *arguments* is

$$40^0,\ 41^0,\ 42^0,\ 43^0;$$

and the given series of corresponding *functions* is

$$45.96\,m.,\ 45.28\,m.,\ 44.59\,m.,\ 43.88\,m.$$

The term to be interpolated in the latter series, is a function of 42° 30′; and this number, in the first series, would be removed $2\frac{1}{2}$ *intervals* from the first term 40°.

Hence, the value of y in the Formula, in the present case, is $2\frac{1}{2}$.

The first terms of the successive *orders of differences* in the series of functions, will be found to be

$$D'=-.68;\ D''=-.01;\ D'''=-.01.$$

Hence, by making the proper substitutions for a, y, &c., in the Formula—omitting D''', since its product would have no significant figures in the first or second decimal places—the required number is

$$45.96-2\tfrac{1}{2}\times .68-\frac{2\frac{1}{2}\times 1\frac{1}{2}}{2}\times .01=44.24.$$

We have thus found that, in latitude 42° 30′, the length of one degree of longitude is 44.24 miles.

When the values of D', D'', &c., do not terminate in 0, the accuracy of the interpolation will depend on the number of these terms that are used; but for most practical purposes it will be sufficient to employ only D' and D'' in the computation.

This will depend, however, on the comparative value of D''', and whether this value will be sensibly increased or diminished by its multiplier in the operation.

When the term to be interpolated is one of the equidistant Functions.

(366.) Whenever D''', or D'''', &c., may, without important error, be taken at 0; the value of one of the intermediate *equidistant* Functions, may be found from the Equation, which is formed by putting the value of D''', or D'''', &c., as expressed in terms of the *functions* (362), equal to 0.

Suppose it were required the number of miles in a degree of longitude, in latitude 42°, knowing that in latitude 40° a degree of longitude is 45.96 miles; in latitude 41°, 45.28 miles; and in latitude 43°, 43.88 miles.

The value of D''', as found under proposition (362), is

$$d-3c+3b-a;$$

a, b, c, d, representing the successive terms of the given Series, which we will suppose to be the Functions

$$45.96\ m.,\ 45.28\ m. \dots 43.88\ m.,$$

in the present example, the third term c, being the one which is to be interpolated.

Taking the value of D''' at 0, we have

$$d-3c+3b-a=0;$$

which gives $c=\dfrac{3b+d-a}{3}=\dfrac{3\times45.28+43.88-45.96}{3}=44.586.$

The value of D''', or D'''', &c., cannot be determined without the term which it is required to find. But the series to which interpolation is usually applied, are themselves not strictly accurate; and as the successive orders of differences rapidly tend to 0, we may generally, for practical purposes, suppose D''', or D'''', to be 0.

EXERCISES

On the Interpolation of Series.

1. Find, by interpolation, the square root of $24\frac{3}{4}$; having given

$$\sqrt{23}=4.79583;\ \sqrt{24}=4.89898;\ \sqrt{25}=5.$$

In the series of numbers 23, 24, 25, it is plain that $24\frac{3}{4}$ is at the distance of $1\frac{3}{4}$ *intervals* from the first term. The value of y in the Formula (365), is therefore $1\frac{3}{4}$.

Ans. 4.9749

2. Find, by interpolation, the cube root of $66\frac{1}{2}$; having given

$$\sqrt[3]{64}=4;\ \sqrt[3]{65}=4.0207;\ \sqrt[3]{66}=4{,}0412;\ \sqrt[3]{67}=4.0615.$$

Ans. 4.0514.

3. Find, by interpolation, the logarithm of 103; having given

the logarithm of 101 = 2.004321;
the logarithm of 102 = 2.008600;
the logarithm of 104 = 2.017033; (366).

Ans. 2.012837.

4. Find, by interpolation, the logarithm of 587; having given

the logarithm of 585 = 2.767156;
the logarithm of 586 = 2.767898;
the logarithm of 588 = 2.769377.

Ans. 2.768638.

5. Find the amount of $1 at compound interest, at 6 per cent, for 12 years and 3 months; knowing that the amount for 10 years is $1.79084, for 11 years $1.89829, for 12 years $2.01219, and for 13 years $2.13293. *Ans.* $2. 04167.

6. Find the number of miles in one degree of longitude, in latitude 38° 15′; knowing that, in latitude 36°, a degree of longitude is 48.54 miles; in latitude 37°, 47.92 miles; in latitude 38°, 47.28 miles; and in latitude 39°, 46.63 miles. *Ans.* 47.12 miles.

A TABLE

OF

LOGARITHMS OF NUMBERS

FROM 1 TO 10,000.

N.	Log.	N.	Log.	N.	Log.	N.	Log.
1	0·000000	26	1·414973	51	1·707570	76	1·880814
2	0·301030	27	1·431364	52	1·716003	77	1·886491
3	0·477121	28	1·447158	53	1·724276	78	1·892095
4	0·602060	29	1·462398	54	1·732394	79	1·897627
5	0·698970	30	1·477121	55	1·740363	80	1·903090
6	0·778151	31	1·491362	56	1·748188	81	1·908485
7	0·845098	32	1·505150	57	1·755875	82	1·913814
8	0·903090	33	1·518514	58	1·763428	83	1·919078
9	0·954243	34	1·531479	59	1·770852	84	1·924279
10	1·000000	35	1·544068	60	1·778151	85	1·929419
11	1·041393	36	1·556303	61	1·785330	86	1·934498
12	1·079181	37	1·568202	62	1·792392	87	1·939519
13	1·113943	38	1·579784	63	1·799341	88	1·944483
14	1·146128	39	1·591065	64	1·806181	89	1·949390
15	1·176091	40	1·602060	65	1·812913	90	1·954243
16	1·204120	41	1·612784	66	1·819544	91	1·959041
17	1·230449	42	1·623249	67	1·826075	92	1·963788
18	1·255273	43	1·633468	68	1·832509	93	1·968483
19	1·278754	44	1·643453	69	1·838849	94	1·973128
20	1·301030	45	1·653213	70	1·845098	95	1·977724
21	1·322219	46	1·662758	71	1·851258	96	1·982271
22	1·342423	47	1·672098	72	1·857333	97	1·986772
23	1·361728	48	1·681241	73	1·863323	98	1·991226
24	1·380211	49	1·690196	74	1·869232	99	1·995635
25	1·397940	50	1·698970	75	1·875061	100	2·000000

Remark. In the following table, in the nine right hand columns of each page, where the first or leading figures change from 9's to 0's, points or dots are introduced instead of the 0's, to catch the eye, and to indicate that from thence the two figures of the Logarithm to be taken from the second column, stand in the next line below.

A TABLE OF LOGARITHMS FROM 1 TO 10,000.

N.	0	1	2	3	4	5	6	7	8	9	D.
100	000000	0434	0868	1301	1734	2166	2598	3029	3461	3891	432
101	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174	428
102	8600	9026	9451	9876	•300	•724	1147	1570	1993	2415	424
103	012837	3259	3680	4100	4521	4940	5360	5779	6197	6616	419
104	7033	7451	7868	8284	8700	9116	9532	9947	•361	•775	416
105	021189	1603	2016	2428	2841	3252	3664	4075	4486	4896	412
106	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978	408
107	9384	9789	•195	•600	1004	1408	1812	2216	2619	3021	404
108	033424	3826	4227	4628	5029	5430	5830	6230	6629	7028	400
109	7426	7825	8223	8620	9017	9414	9811	•207	•602	•998	396
110	041393	1787	2182	2576	2969	3362	3755	4148	4540	4932	393
111	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830	389
112	9218	9606	9993	•380	•766	1153	1538	1924	2309	2694	386
113	053078	3463	3846	4230	4613	4996	5378	5760	6142	6524	382
114	6905	7286	7666	8046	8426	8805	9185	9563	9942	•320	379
115	060698	1075	1452	1829	2206	2582	2958	3333	3709	4083	376
116	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815	372
117	8186	8557	8928	9298	9668	••38	•407	•776	1145	1514	369
118	071882	2250	2617	2985	3352	3718	4085	4451	4816	5182	366
119	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819	363
120	079181	9543	9904	•266	•626	•987	1347	1707	2067	2426	360
121	082785	3144	3503	3861	4219	4576	4934	5291	5647	6004	357
122	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552	355
123	9905	•258	•611	•963	1315	1667	2018	2370	2721	3071	351
124	093422	3772	4122	4471	4820	5169	5518	5866	6215	6562	349
125	6910	7257	7604	7951	8298	8644	8990	9335	9681	••26	346
126	100371	0715	1059	1403	1747	2091	2434	2777	3119	3462	343
127	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871	340
128	7210	7549	7888	8227	8565	8903	9241	9579	9916	•253	338
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326	3218	3351	3484	3617	3750	3883	4016	4149	4282	4414	133
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357	2668	2790	2911	3033	3155	3276	3398	3519	3640	3762	121
358	3883	4004	4126	4247	4368	4489	4610	4731	4852	4973	121
359	5094	5215	5336	5457	5578	5699	5820	5940	6061	6182	121
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361	7507	7627	7748	7868	7988	8108	8228	8349	8469	8589	120
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363	9907	••26	•146	•265	•385	•504	•624	•743	•863	•982	119
364	561101	1221	1340	1459	1578	1698	1817	1936	2055	2174	119
365	2293	2412	2531	2650	2769	2887	3006	3125	3244	3362	119
366	3481	3600	3718	3837	3955	4074	4192	4311	4429	4548	119
367	4666	4784	4903	5021	5139	5257	5376	5494	5612	5730	118
368	5848	5966	6084	6202	6320	6437	6555	6673	6791	6909	118
369	7026	7144	7262	7379	7497	7614	7732	7849	7967	8084	118
370	568202	8319	8436	8554	8671	8788	8905	9023	9140	9257	117
371	9374	9491	9608	9725	9842	9959	••76	•193	•309	•426	117
372	570543	0660	0776	0893	1010	1126	1243	1359	1476	1592	117
373	1709	1825	1942	2058	2174	2291	2407	2523	2639	2755	116
374	2872	2988	3104	3220	3336	3452	3568	3684	3800	3915	116
375	4031	4147	4263	4379	4494	4610	4726	4841	4957	5072	116
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378	7492	7607	7722	7836	7951	8066	8181	8295	8410	8525	115
379	8639	8754	8868	8983	9097	9212	9326	9441	9555	9669	114
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385	5461	5574	5686	5799	5912	6024	6137	6250	6362	6475	113
386	6587	6700	6812	6925	7037	7149	7262	7374	7486	7599	112
387	7711	7823	7935	8047	8160	8272	8384	8496	8608	8720	112
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393	4393	4503	4614	4724	4834	4945	5055	5165	5276	5386	110
394	5496	5606	5717	5827	5937	6047	6157	6267	6377	6487	110
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396	7695	7805	7914	8024	8134	8243	8353	8462	8572	8681	110
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398	9883	9992	•101	•210	•319	•428	•537	•646	•755	•864	109
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410	612784	2890	2996	3102	3207	3313	3419	3525	3630	3736	106
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423	6340	6443	6546	6648	6751	6853	6956	7058	7161	7263	103
424	7366	7468	7571	7673	7775	7878	7980	8082	8185	8287	102
425	8389	8491	8593	8695	8797	8900	9002	9104	9206	9308	102
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427	630428	0530	0631	0733	0835	0936	1038	1139	1241	1342	102
428	1444	1545	1647	1748	1849	1951	2052	2153	2255	2356	101
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444	7383	7481	7579	7676	7774	7872	7969	8067	8165	8262	98
445	8360	8458	8555	8653	8750	8848	8945	9043	9140	9237	97
446	9335	9432	9530	9627	9724	9821	9919	••16	•113	•210	97
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468	670246	0339	0431	0524	0617	0710	0802	0695	0988	1080	93
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470	672098	2190	2283	2375	2467	2560	2652	2744	2836	2929	92
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475	6694	6785	6876	6968	7059	7151	7242	7333	7424	7516	91
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481	2145	2235	2326	2416	2506	2596	2686	2777	2867	2957	90
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498	7229	7317	7404	7491	7578	7665	7752	7839	7926	8014	87
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506	4151	4236	4322	4408	4494	4579	4665	4751	4837	4922	86
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508	5864	5949	6035	6120	6206	6291	6376	6462	6547	6632	85
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540	732394	2474	2555	2635	2715	2796	2876	2956	3037	3117	80
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547	7987	8067	8146	8225	8305	8384	8463	8543	8622	8701	79
548	8781	8860	8939	9018	9097	9177	9256	9335	9414	9493	79
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552	1939	2018	2096	2175	2254	2332	2411	2489	2568	2647	79
553	2725	2804	2882	2961	3039	3118	3196	3275	3353	3431	78
554	3510	3588	3667	3745	3823	3902	3980	4058	4136	4215	78
555	4293	4371	4449	4528	4606	4684	4762	4840	4919	4997	78
556	5075	5153	5231	5309	5387	5465	5543	5621	5699	5777	78
557	5855	5933	6011	6089	6167	6245	6323	6401	6479	6556	78
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559	7412	7489	7567	7645	7722	7800	7878	7955	8033	8110	78
560	748188	8266	8343	8421	8498	8576	8653	8731	8808	8885	77
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562	9736	9814	9891	9968	••45	•123	•200	•277	•354	•431	77
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565	2048	2125	2202	2279	2356	2433	2509	2586	2663	2740	77
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572	7396	7472	7548	7624	7700	7775	7851	7927	8003	8079	76
573	8155	8230	8306	8382	8458	8533	8609	8685	8761	8836	76
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585	7156	7230	7304	7379	7453	7527	7601	7675	7749	7823	74
586	7898	7972	8046	8120	8194	8268	8342	8416	8490	8564	74
587	8638	8712	8786	8860	8934	9008	9082	9156	9230	9303	74
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605	1755	1827	1899	1971	2042	2114	2186	2258	2329	2401	72
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633	1404	1472	1541	1609	1678	1747	1815	1884	1952	2021	69
634	2089	2158	2226	2295	2363	2432	2500	2568	2637	2705	69
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685	5691	5754	5817	5881	5944	6007	6071	6134	6197	6261	63
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745	2156	2215	2273	2331	2389	2448	2506	2564	2622	2681	58
746	2739	2797	2855	2913	2972	3030	3088	3146	3204	3262	58
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779	1537	1593	1649	1705	1760	1816	1872	1928	1983	2039	56
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781	2651	2707	2762	2818	2873	2929	2985	3040	3096	3151	56
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791	8176	8231	8286	8341	8396	8451	8506	8561	8615	8670	55
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797	1458	1513	1567	1622	1676	1731	1785	1840	1894	1948	54
798	2003	2057	2112	2166	2221	2275	2329	2384	2438	2492	54
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801	3633	3687	3741	3795	3849	3904	3958	4012	4066	4120	54
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805	5796	5850	5904	5958	6012	6066	6119	6173	6227	6281	54
806	6335	6389	6443	6497	6551	6604	6658	6712	6766	6820	54
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811	9021	9074	9128	9181	9235	9289	9342	9396	9449	9503	54
812	9556	9610	9663	9716	9770	9823	9877	9930	9984	••37	53
813	910091	0144	0197	0251	0304	0358	0411	0464	0518	0571	53
814	0624	0678	0731	0784	0838	0891	0944	0998	1051	1104	53
815	1158	1211	1264	1317	1371	1424	1477	1530	1584	1637	53
816	1690	1743	1797	1850	1903	1956	2009	2063	2116	2169	53
817	2222	2275	2328	2381	2435	2488	2541	2594	2647	2700	53
818	2753	2806	2859	2913	2966	3019	3072	3125	3178	3231	53
819	3284	3337	3390	3443	3496	3549	3602	3655	3708	3761	53

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820	913814	3867	3920	3973	4026	4079	4132	4184	4237	4290	53
821	4343	4396	4449	4502	4555	4608	4660	4713	4766	4819	53
822	4872	4925	4977	5030	5083	5136	5189	5241	5294	5347	53
823	5400	5453	5505	5558	5611	5664	5716	5769	5822	5875	53
824	5927	5980	6033	6085	6138	6191	6243	6296	6349	6401	53
825	6454	6507	6559	6612	6664	6717	6770	6822	6875	6927	53
826	6980	7033	7085	7138	7190	7243	7295	7348	7400	7453	53
827	7506	7558	7611	7663	7716	7768	7820	7873	7925	7978	52
828	8030	8083	8135	8188	8240	8293	8345	8397	8450	8502	52
829	8555	8607	8659	8712	8764	8816	8869	8921	8973	9026	52
830	919078	9130	9183	9235	9287	9340	9392	9444	9496	9549	52
831	9601	9653	9706	9758	9810	9862	9914	9967	●●19	●●71	52
832	920123	0176	0228	0280	0332	0384	0436	0489	0541	0593	52
833	0645	0697	0749	0801	0853	0906	0958	1010	1062	1114	52
834	1166	1218	1270	1322	1374	1426	1478	1530	1582	1634	52
835	1686	1738	1790	1842	1894	1946	1998	2050	2102	2154	52
836	2206	2258	2310	2362	2414	2466	2518	2570	2622	2674	52
837	2725	2777	2829	2881	2933	2985	3037	3089	3140	3192	52
838	3244	3296	3348	3399	3451	3503	3555	3607	3658	3710	52
839	3762	3814	3865	3917	3969	4021	4072	4124	4176	4228	52
840	924279	4331	4383	4434	4486	4538	4589	4641	4693	4744	52
841	4796	4848	4899	4951	5003	5054	5106	5157	5209	5261	52
842	5312	5364	5415	5467	5518	5570	5621	5673	5725	5776	52
843	5828	5879	5931	5982	6034	6085	6137	6188	6240	6291	51
844	6342	6394	6445	6497	6548	6600	6651	6702	6754	6805	51
845	6857	6908	6959	7011	7062	7114	7165	7216	7268	7319	51
846	7370	7422	7473	7524	7576	7627	7678	7730	7781	7832	51
847	7883	7935	7986	8037	8088	8140	9191	8242	8293	8345	51
848	8396	8447	8498	8549	8601	8652	8703	8754	8805	8857	51
849	8908	8959	9010	9061	9112	9163	9215	9266	9317	9368	51
850	929419	9470	9521	9572	9623	9674	9725	9776	9827	9879	51
851	9930	9981	●●32	●●83	●134	●185	●236	●287	●338	●389	51
852	930440	0491	0542	0592	0643	0694	0745	0796	0847	0898	51
853	0949	1000	1051	1102	1153	1204	1254	1305	1356	1407	51
854	1458	1509	1560	1610	1661	1712	1763	1814	1865	1915	51
855	1966	2017	2068	2118	2169	2220	2271	2322	2372	2423	51
856	2474	2524	2575	2626	2677	2727	2778	2829	2879	2930	51
857	2981	3031	3082	3133	3183	3234	3285	3335	3386	3437	51
858	3487	3538	3589	3639	3690	3740	3791	3841	3892	3943	51
859	3993	4044	4094	4145	4195	4246	4296	4347	4397	4448	51
860	934498	4549	4599	4650	4700	4751	4801	4852	4902	4953	50
861	5003	5054	5104	5154	5205	5255	5306	5356	5406	5457	50
862	5507	5558	5608	5658	5709	5759	5809	5860	5910	5960	50
863	6011	6061	6111	6162	6212	6262	6313	6363	6413	6463	50
864	6514	6564	6614	6665	6715	6765	6815	6865	6916	6966	50
865	7016	7066	7117	7167	7217	7267	7317	7367	7418	7468	50
866	7518	7568	7618	7668	7718	7769	7819	7869	7919	7969	50
867	8019	8069	8119	8169	8219	8269	8320	8370	8420	8470	50
868	8520	8570	8620	8670	8720	8770	8820	8870	8920	8970	50
869	9020	9070	9120	9170	9220	9270	9320	9369	9419	9469	50
870	939519	9569	9619	9669	9719	9769	9819	9869	9918	9968	50
871	940018	0068	0118	0168	0218	0267	0317	0367	0417	0467	50
872	0516	0566	0616	0666	0716	0765	0815	0865	0915	0964	50
873	1014	1064	1114	1163	1213	1263	1313	1362	1412	1462	50
874	1511	1561	1611	1660	1710	1760	1809	1859	1909	1958	50
875	2008	2058	2107	2157	2207	2256	2306	2355	2405	2455	50
876	2504	2554	2603	2653	2702	2752	2801	2851	2901	2950	50
877	3000	3049	3099	3148	3198	3247	3297	3346	3396	3445	49
878	3495	3544	3593	3643	3692	3742	3791	3841	3890	3939	49
879	3989	4038	4088	4137	4186	4236	4285	4335	4384	4433	49

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880	944483	4532	4581	4631	4680	4729	4779	4828	4877	4927	49
881	4976	5025	5074	5124	5173	5222	5272	5321	5370	5419	49
882	5469	5518	5567	5616	5665	5715	5764	5813	5862	5912	49
883	5961	6010	6059	6108	6157	6207	6256	6305	6354	6403	49
884	6452	6501	6551	6600	6649	6698	6747	6796	6845	6894	49
885	6943	6992	7041	7090	7140	7189	7238	7287	7336	7385	49
886	7434	7483	7532	7581	7630	7679	7728	7777	7826	7875	49
887	7924	7973	8022	8070	8119	8168	8217	8266	8315	8364	49
888	8413	8462	8511	8560	8609	8657	8706	8755	8804	8853	49
889	8902	8951	8999	9048	9097	9146	9195	9244	9292	9341	49
890	949390	9439	9488	9536	9585	9634	9683	9731	9780	9829	49
891	9878	9926	9975	••24	••73	•121	•170	•219	•267	•316	49
892	950365	0414	0462	0511	0560	0608	0657	0706	0754	0803	49
893	0851	0900	0949	0997	1046	1095	1143	1192	1240	1289	49
894	1338	1386	1435	1483	1532	1580	1629	1677	1726	1775	49
895	1823	1872	1920	1969	2017	2066	2114	2163	2211	2260	48
896	2308	2356	2405	2453	2502	2550	2599	2647	2696	2744	48
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899	3760	3808	3856	3905	3953	4001	4049	4098	4146	4194	48
900	954243	4291	4339	4387	4435	4484	4532	4580	4628	4677	48
901	4725	4773	4821	4869	4918	4966	5014	5062	5110	5158	48
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906	7128	7176	7224	7272	7320	7368	7416	7464	7512	7559	48
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909	8564	8612	8659	8707	8755	8803	8850	8898	8946	8994	48
910	959041	9089	9137	9185	9232	9280	9328	9375	9423	9471	48
911	9518	9566	9614	9661	9709	9757	9804	9852	9900	9947	48
912	9995	••42	••90	•138	•185	•233	•280	•328	•376	•423	48
913	960471	0518	0566	0613	0661	0709	0756	0804	0851	0899	48
914	0946	0994	1041	1089	1136	1184	1231	1279	1326	1374	47
915	1421	1469	1516	1563	1611	1658	1706	1753	1801	1848	47
916	1895	1943	1990	2038	2085	2132	2180	2227	2275	2322	47
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920	963788	3835	3882	3929	3977	4024	4071	4118	4165	4212	47
921	4260	4307	4354	4401	4448	4495	4542	4590	4637	4684	47
922	4731	4778	4825	4872	4919	4966	5013	5061	5108	5155	47
923	5202	5249	5296	5343	5390	5437	5484	5531	5578	5625	47
924	5672	5719	5766	5813	5860	5907	5954	6001	6048	6095	47
925	6142	6189	6236	6283	6329	6376	6423	6470	6517	6564	47
926	6611	6658	6705	6752	6799	6845	6892	6939	6986	7033	47
927	7080	7127	7173	7220	7267	7314	7361	7408	7454	7501	47
928	7548	7595	7642	7688	7735	7782	7829	7875	7922	7969	47
929	8016	8062	8109	8156	8203	8249	8296	8343	8390	8436	47
930	968483	8530	8576	8623	8670	8716	8763	8810	8856	8903	47
931	8950	8996	9043	9090	9136	9183	9229	9276	9323	9369	47
932	9416	9463	9509	9556	9602	9649	9695	9742	9789	9835	47
933	9882	9928	9975	••21	••68	•114	•161	•207	•254	•300	47
934	970347	0393	0440	0486	0533	0579	0626	0672	0719	0765	46
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936	1276	1322	1369	1415	1461	1508	1554	1601	1647	1693	46
937	1740	1786	1832	1879	1925	1971	2018	2064	2110	2157	46
938	2203	2249	2295	2342	2388	2434	2481	2527	2573	2619	46
939	2666	2712	2758	2804	2851	2897	2943	2989	3035	3082	46

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940	973128	3174	3220	3266	3313	3359	3405	3451	3497	3543	46
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942	4051	4097	4143	4189	4235	4281	4327	4374	4420	4466	46
943	4512	4558	4604	4650	4696	4742	4788	4834	4880	4926	46
944	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386	46
945	5432	5478	5524	5570	5616	5662	5707	5753	5799	5845	46
946	5891	5937	5983	6029	6075	6121	6167	6212	6258	6304	46
947	6350	6396	6442	6488	6533	6579	6625	6671	6717	6763	46
948	6808	6854	6900	6946	6992	7037	7083	7129	7175	7220	46
949	7266	7312	7358	7403	7449	7495	7541	7586	7632	7678	46
950	977724	7769	7815	7861	7906	7952	7998	8043	8089	8135	46
951	8181	8226	8272	8317	8363	8409	8454	8500	8546	8591	46
952	8637	8683	8728	8774	8819	8865	8911	8956	9002	9047	46
953	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503	46
954	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958	46
955	980003	0049	0094	0140	0185	0231	0276	0322	0367	0412	45
956	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867	45
957	0912	0957	1003	1048	1093	1139	1184	1229	1275	1320	45
958	1366	1411	1456	1501	1547	1592	1637	1683	1728	1773	45
959	1819	1864	1909	1954	2000	2045	2090	2135	2181	2226	45
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961	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130	45
962	3175	3220	3265	3310	3356	3401	3446	3491	3536	3581	45
963	3626	3671	3716	3762	3807	3852	3897	3942	3987	4032	45
964	4077	4122	4167	4212	4257	4302	4347	4392	4437	4482	45
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970	986772	6817	6861	6906	6951	6996	7040	7085	7130	7175	45
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972	7666	7711	7756	7800	7845	7890	7934	7979	8024	8068	45
973	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514	45
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975	9005	9049	9094	9138	9183	9227	9272	9316	9361	9405	45
976	9450	9494	9539	9583	9628	9672	9717	9761	9806	9850	44
977	9895	9939	9983	●●28	●●72	●117	●161	●206	●250	●294	44
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982	2111	2156	2200	2244	2288	2333	2377	2421	2465	2509	44
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984	2995	3039	3083	3127	3172	3216	3260	3304	3348	3392	44
985	3436	3480	3524	3568	3613	3657	3701	3745	3789	3833	44
986	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273	44
987	4317	4361	4405	4449	4493	4537	4581	4625	4669	4713	44
988	4757	4801	4845	4889	4933	4977	5021	5065	5108	5152	44
989	5196	5240	5284	5328	5372	5416	5460	5504	5547	5591	44
990	995635	5679	5723	5767	5811	5854	5898	5942	5986	6030	44
991	6074	6117	6161	6205	6249	6293	6337	6380	6424	6468	44
992	6512	6555	6599	6643	6687	6731	6774	6818	6862	6906	44
993	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343	44
994	7386	7430	7474	7517	7561	7605	7648	7692	7736	7779	44
995	7823	7867	7910	7954	7998	8041	8085	8129	8172	8216	44
996	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652	44
997	8695	8739	8782	8826	8869	8913	8956	9000	9043	9087	44
998	9131	9174	9218	9261	9305	9348	9392	9435	9479	9522	44
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